Experiment 1
Measuring “g” using the Kater Pendulum

Goal:

The free fall acceleration, “g”, is a quantity that varies by location around the surface of the earth. While everywhere on the surface it is 9.8 m/s² to two significant digits, for some experiments we will need to have more precision. Moreover, it will be important to know the uncertainty in the local value of g. The purpose of this experiment is to perform a careful measurement of the local value of g that can then be used in calculations in future experiments.

Background:

The experiment will involve the use of a physical pendulum. Recall that the difference between a simple pendulum and a physical pendulum is that a simple pendulum has all of its mass concentrated at the tip while a physical pendulum has mass distributed throughout its length.

Consider an arbitrary pendulum, as indicated above. The key characteristics are the distance between the center of mass and the pivot point (“d” in the above diagram), the moment of inertia (I) about the pivot point, and the total mass of the pendulum, M. If you consult an introductory physics textbook or Analytical Mechanics by Fowles and Cassiday, you will find that the period for such a pendulum is given by

\[ T = 2\pi \sqrt{\frac{I}{Mgd}} \]

(1)

This equation applies provided the amplitude of oscillation is sufficiently small.
Note that in the case of a simple pendulum, the string or rod is taken to be massless so that $I = Md^2$, reducing (1) to

$$T = 2\pi \sqrt{\frac{d}{g}}$$

(2)

This commonly appears in textbooks with “d” replaced by “L”. These two variables have the same meaning in the context of the simple pendulum. A quick way to determine $g$, then, is to measure the period of a simple pendulum and rearrange (2) to obtain:

$$g = 4\pi^2 \frac{d}{T^2}$$

(3)

This method will be explored briefly in this experiment, long enough to convince you that the Kater pendulum approach is better.

Returning to the physical pendulum as described by equation (1), the moment of inertia of an object depends on the location of the axis of rotation (or the pivot point in our case). The Parallel Axis Theorem says that the moment of inertia, $I$, about an axis parallel to one that passes through the center of mass of an object is given by

$$I = I_{cm} + Md^2$$

(4)

where $I_{cm}$ is the moment of inertia about the center of mass and $d$ is the distance between the center of mass and the axis of rotation. It is convenient to write the moment of inertia about the center of mass in terms of $k$, a quantity known as the radius of gyration:

$$I_{cm} = Mk^2$$

(5)

Here $k$ represents the distance from the axis all the mass could be concentrated such that the given moment of inertia is reproduced.

Combining the previous equations (1), (4), and (5) gives us

$$T = 2\pi \sqrt{\frac{k^2 + d^2}{gd}}$$

(6)

Now let us ask, are there two different pivot points we could use to produce the same period? When we change the pivot point, only the value of $d$ in the above equation changes ($k$ is determined exclusively by the moment of inertia about the center of mass). We will use subscripts 1 and 2 to refer to the two different pivot points:
A little bit of algebra reduces this to the constraint

\[ k^2 = d_1 d_2 \]  \hspace{1cm} (7)

We will call this common value of the period we obtain when the above relation is satisfied, \( T^* \). Using the equation for \( T_1 \), we find

\[ T^* = 2\pi \sqrt{\frac{k^2 + d_1^2}{gd}} = 2\pi \sqrt{\frac{d_1 d_2 + d_2^2}{gd}} = 2\pi \sqrt{\frac{d_1 + d_2}{g}} \]

Of course our goal in this experiment is to find \( g \):

\[ g = \frac{4\pi^2 (d_1 + d_2)}{(T^*)^2} \]  \hspace{1cm} (8)

Thus our value of \( g \) will depend on

- our ability to find two different pivots on the same pendulum that produce the same period,
- our ability to measure precisely the distance between these two pivot points,
- our ability to measure precisely the pendulum period.

Had we instead elected to work with the original equation for the period (1), we would have been faced with the need to determine the position of the center of mass with precision as well as the moment of inertia about the pivot point with precision. While these calculations are easy for idealized situations, getting precise values for a real pendulum (with, for instance, a hole punched into it to allow for a pivot point) is not trivial.
**Procedure:**

**Part 1:**

The apparatus for this part consists of a stopwatch, a rubber stopper, some thread, and a tall support stand. Use the rod that runs perpendicular to the wall—leave the rubber stopper on the end (it is there to keep the rod from poking people in the eye).

1. Your first task is to make a 50 cm long simple pendulum. If there is already a thread of the right length attached to a stopper, skip to step 2. Cut a piece of thread approximately 75 cm long. Thread one end of it through the hole in a free rubber stopper and tie it off by looping it back above the stopper. Tie a large loop in the other end of the thread so that the pendulum length—as measured from the point at which the pendulum will pivot to the center of mass of the stopper—is about 50 cm.

2. Rotate the thin rod (on the right hand side of the stand) down so that the rubber-tipped end points towards you and the rod is horizontal. Suspend the pendulum from the large stand and then measure the distance from the pivot point to the center of mass of the stopper, as carefully as you can, with a meter stick. Estimate the uncertainty in this measured length, noting that the uncertainty associated with the procedure is likely greater than the uncertainty associated with the instrument.

To help understand sources of uncertainty and ways to reduce them, you will next measure the period in two different ways. Please follow the procedure as described, even though you may recognize the first approach as crude.

3. Approach A: Pull the pendulum back 10 cm or so (this distance is not critical) and release it. Using a stopwatch, time one cycle of the oscillation. You can, for instance, start the stopwatch when the pendulum is on the far left and stop it when it returns to that side. **DO NOT** start the stopwatch when the pendulum is released. You are more likely to introduce systematic error that way (see the discussion in the Data Analysis section below). A technique that is probably more accurate is to time the pendulum as it passes through the midpoint of the oscillation (you can use the meter stick taped to the wall to line this up). Start the watch when it passes through coming from the left, and stop it when it comes back through the midpoint from the left again. Since this pendulum is moving fast through the center, you have a shorter “triggering” event than if you were to use the end point of the cycle (when the pendulum moves slowly) to time the period. Make and record a total of 10 separate, single cycle measurements of the period. You do not need to restart the oscillation each time.

4. Approach B: This process is the same as the previous one except you will time 10 consecutive cycles. That is, start the stopwatch when the pendulum passes through the midpoint coming from the left. The first cycle is complete the next time it comes through from the left, and so on. After 10 cycles have been completed, stop the watch and record the total time. Repeat this 10 times, for a total of ten measurements of ten cycles.
Part 2:

The apparatus for this part consists of a brass rod (the pendulum) with two holes drilled in it (for the two different pivot points), a plastic slide that can be moved up and down the rod (allowing you to change the pendulum’s mass distribution), and a precision electronic timer.

I will refer to the hole near the end of the rod as the “end hole” and the one closer to the center as the “mid hole”. The rod has lines etched in it at 1 cm intervals, except where the line would be obscured by a hole. I will refer to the end opposite to the end hole as the zero point on the slider position scale. The diagram below should clarify this:

![Diagram of mid and end holes with 0, 1, 2, 3, 4... labels]

1. Place the slider at the 8 cm mark. **This is a precision experiment. Do not place tape or any other type of markings on the brass pendulum.** Note that there are not many adjustments to make in this experiment, but each one must be made with care in order to produce accurate results. Mount the pendulum on the left side of the stand, on the “knife-edge”, which is actually a short rod with a square cross section, oriented so that one corner of the square is at the top. Start by mounting the pendulum from the end hole. Do not mount the pendulum all the way at the tip of the knife edge; rather move it towards the support rod by a centimeter or so, because later you will need some room to slide it out towards the tip. **Make sure that the knife-edge is centered in the hole as closely as possible.** This turns out to be a critical step in order to get reproducible results. You can use the provided magnifying glass to make sure the contact point between the knife-edge and the top of the hole is in line with the engraved dots that run down the center of the pendulum. This centering check needs to be performed each time you remount the pendulum.

2. Test the swing for stability by setting up a swing with a roughly 0.5 cm amplitude (i.e., a total swing of about 1 cm). Note that the actual value of the amplitude is not relevant, but being relatively close to 0.5 cm will be necessary to get the photogate to function properly. If the pendulum is mounted correctly, the oscillations should settle down into a planar mode. If the rod never stops twisting or circling, then either the knife-edge is not centered in the hole or the support mechanism is not level (in the latter case, you may need to shim the tripod with some paper).

3. Next mount the photogate so that the “U” is parallel to the plane of the table and such that the pendulum swings though the U, parallel to the sides of the U and perpendicular to...
the base of the U. Let the pendulum hang vertically, just barely touching the inner edge of the far side of the U. You will notice that on the inner surface near each tip of the U there is a small circular hole. These holes comprise the photogate. The photogate detects interruption of the light signal between these two points. Your goal is to line up the photogate with the point marked A on the diagram below:

![Diagram](image)

It seems to be easier to make this adjustment when the pendulum is resting against one side of the U. Once you have the photogate correctly positioned, carefully slide the pendulum along the knife-edge away from the photogate so that when it swings, it will not rub the photogate. You will need about a 1 cm gap between the rod and the photogate, so that later on in the experiment there is room for the slider at the photogate height. As always, make sure that the knife-edge remains centered in the pivot hole following the procedure in step 1. Make a final check to ensure the pendulum swings freely and without any twisting motion.

4. There should be a cable running from the photogate to the back of the electronic timer, plugged into the ΔT port. Turn the electronic timer on. The Mode switch should be slid to the right, so that the LED next to the Period label (actually slightly above it) is lit. Once the pendulum is set in motion, the readout will display the period, updating it about every 10 s. Displace the pendulum about 0.5 cm to one side and release it. If the alignment is correct and the amplitude is correct, the pendulum will swing so that point B momentarily blocks the photogate, but point B does not continue on past the photogate. When the swing settles down, you should get a period very close to (i.e., within a hundredths of a second), but not exactly equal to, one second.

Troubleshooting tips:

- The pendulum continues to wobble: Try shifting to a slightly different position on the knife-edge, check to make sure the knife edge is centered in the hole, make sure your amplitude is not too large, or be more careful how you release the pendulum.

- The period reads close to 2 s rather than 1 s. Most likely, either the amplitude is too small or the electric eye has been lined up with the hole instead of at point A.
• The period is too small: Try using a smaller amplitude or waiting longer for the wobbles to die out.

If the period is right around one second and you have a nice stable swing, then the alignment is complete. Remaining adjustments should be easy and minimal, **provided you do not bump the apparatus.**

5. To take data, let the pendulum swing until the period settles down. Avoid touching the table during this time. Watch the Period readout as it updates every 10 seconds. If there are four consecutive readings all within 0.0004 s of each other, then the oscillation is sufficiently stable, so you can record the average of those 4 times (to the nearest 0.0001 s). I have found that this stability criterion is sufficient to generate reproducible results on almost all of my trials. This period corresponds to pivoting about the end hole with the slider at the 8 cm position. If you are having trouble getting the oscillations to settle down, check to see that the pendulum is properly centered on the knife-edge.

6. Now carefully remove the pendulum from the knife-edge and flip it over, remounting it in the mid hole. Check for centering following the procedure in step 1. Measure the period as you did in step 5.

7. Move the slider to the 10 cm mark, return the pendulum to the knife-edge, and measure the periods once again for both mounting holes. Repeat for the slider at 12 cm, 14 cm, and 16 cm, so that you will have five positions altogether, each with period measurements at both suspension holes. The last two slider positions are on the other side of the mid hole. Remember that one of the marks on the rod is missing, due to the presence of the mid hole (i.e., that hole counts as a line). Each of the periods should be different, but all should be within 0.01 s of 1.0000 s. In all, you should have recorded the value of 10 different periods. Be careful when the slider is at the 12 cm mark that it does not rub into the photogate (this added noise to my data until I spotted the problem).

8. When you have completed your measurements, take the pendulum off the knife-edge and place it on the table where it is visible for the next group but where it will not easily be knocked off. Make sure to turn off the timer. The only data you need to acquire for this part of the experiment are these periods (two sets of periods, each with five times).
Data Analysis:

Part 1

Approach A: Determine the average value of the ten single-cycle measurements, and then determine the maximum value and the minimum value. While with large data sets it is appropriate to calculate the uncertainty using the standard deviation or the square root of the variance, it is not clear that this approach is appropriate when a small number of measurements are made. For most experiments in this course, we will not be repeating our measurements often enough to make it worthwhile calculating the standard deviation (although you should feel free to do so if you wish). Instead, we will use a cruder approach that gets the order of magnitude right. The uncertainty, $\delta T$, in our measured value of $T$ is the smallest number such that $T + \delta T$ incorporates all of the data. In the following simple case,

\[
\begin{array}{cccccc}
\text{Trial} & 1 & 2 & 3 & 4 & 5 \\
\text{Time (s)} & 1.10 & 1.13 & 1.04 & 1.25 & 1.08 \\
\end{array}
\]

we would calculate $T = 1.12 \pm 0.13$ s since the average time is 1.12 s and all of the data lies in the interval of (0.99s, 1.25s). There will never be any case in this course where you should use more than 2 significant digits in your uncertainty, and usually 1 significant digit is fine. Once you have determined the number of significant digits you will report in the uncertainty, the precision of the computed value would be rounded off to the same level. That is, we could also report the above result as 1.1 $\pm$ 0.1 s (the final result given to the nearest tenth of a second in this case).

Approach B: In this approach you measured a time interval for 10 full cycles. Denote this as $T_{10}$. Divide each of the values of $T_{10}$ by 10 to get the time for one period. Then analyze those times as you did above. You should notice that the uncertainty is smaller in this approach by \textit{about} a factor of 10. In fact, if you review your calculations, you will see how the uncertainty effectively gets reduced by a factor of 10. Notice that when we measure the time for ten cycles to the nearest hundredth of a second and then divide by ten to get the period, this gives us information about the period potentially to the nearest thousandth of a second. Keep this thousandths digit in your calculations until you see if, based on your uncertainty, it is significant.

Why is there uncertainty in this data? The primary reason is that your timing interval depends on your observing the starting points and stopping points correctly and on your ability to physically respond to your observations. We loosely refer to this as a reaction time issue. If your reaction time were always the same, then it would delay the start and stop of the watch in the same way, resulting in no error introduced into the time interval. It is the fact that the reaction time \textit{varies} some that introduces some uncertainty into the data.
Systematic error vs. random error: It is not hard to imagine the possibility that the reaction time for starting the watch and stopping the watch could be slightly different, depending on how the experiment is performed. If the reaction time for starting were always longer than for stopping, that would lead to a systematic error, in this case all of your measured times being too short. This systematic error would not show up in the uncertainty analysis above. That analysis assumed that the timing error was uniformly distributed around the actual time. A careful experimental procedure will minimize systematic error (see Procedure Part 1, Step 3). In some cases, there may be remaining, identifiable systematic error. For instance, equation (3), which is used to analyze this data, is based on a small angle approximation. The approximation used therefore introduces systematic error into calculated results. Ideally, an experimentalist can identify all sources of systematic error and demonstrate that any resulting corrections would be smaller than those introduced by random errors, and hence they do not need to be included in the analysis. The reality is, though, that systematic error often creeps in through poor experimental design and hence may only be spotted by an outside observer. The bottom line is that your goal should be to reduce any sources of systematic error to a level less than that of the random error, so that systematic sources no longer become relevant.

So we will assume that the timing uncertainty is all the product of random, not systematic, error. Next we turn to the length measurement ("d" in equation 3). There are two issues to consider here. First, how precisely can you measure with a meter stick? The smallest unit marked is 1 mm. Some would argue that you can therefore only measure to the nearest mm, others will argue that you can visually detect to the nearest 0.5 mm. My perspective is that either position is justifiable and depends on the person making the measurement. This illustrates the point that there is a role for scientific judgment in performing and analyzing experiments. We are more than machines following rules.

The second issue in determining the uncertainty in d is our ability to accurately locate the pivot point and the center of mass of the stopper. The latter is generally based on an educated guess. It seems likely that the uncertainty in its position exceeds the uncertainty introduced due to the measuring limit of the meter stick. I would estimate this contribution to the uncertainty to be 3-5 mm. In sum, it would be reasonable to attribute an uncertainty of 5 mm to the length measurement.

Notice that in both the time and distance measurements, the uncertainty for the data significantly exceeded the measurement limitation of the device. That will be a common occurrence in this course.
Carrying uncertainty analysis through in your calculations

A word of caution: make sure to retain enough significant digits in your intermediate calculations to support your final result. For instance, in Part 2 of this experiment, you will be calculating g to three or four significant digits, so your intermediate calculations must be good to at least four digits. This means, for instance, you cannot use the value of 3.14 for pi.

Returning to equation (3):

\[ g = 4\pi^2 \frac{d}{T^2} \]

we have seen there is uncertainty in values for both d and T. Once the average (best) values and the uncertainties have been established (as above), then

\[
g_{\text{best}} = 4\pi^2 \frac{d_{\text{best}}}{(T_{\text{best}})^2}
\]

\[
g_{\text{min}} = 4\pi^2 \frac{d_{\text{best}} - \delta d}{(T_{\text{best}} + \delta T)^2}
\]

\[
g_{\text{max}} = 4\pi^2 \frac{d_{\text{best}} + \delta d}{(T_{\text{best}} - \delta T)^2}
\]

Notice that to get the maximum value of g, we maximize the numerator and minimize the denominator. Likewise, to get the minimum value of g possible, we minimize the numerator and maximize the denominator. Your result is then reported as \( g +/\delta g \), where g is the best value and \( \delta g \) is taken just large enough to incorporate both the minimum and maximum values of g. This crude approach to calculating uncertainties usually overstates the uncertainty and thus can be viewed as a cautious estimate of the actual uncertainty. Make sure to follow this analysis through for both approaches in part 1.

Here is an alternate approach to calculating the uncertainty that you learned in PHY 223 and you are welcome to use here. For a function, f, of the form

\[ f = Ax^n y^m, \]

where x and y have uncertainties \( \delta x \) and \( \delta y \) respectively, the uncertainty in f is given by

\[
\delta f = f \left[ \left( n \frac{\delta x}{x} \right)^2 + \left( m \frac{\delta y}{y} \right)^2 \right]^{1/2}
\]

(9)

Applying this to our equation for g,

\[
\delta g = g \left[ \left( \frac{\delta d}{d} \right)^2 + \left( -2 \frac{\delta T}{T} \right)^2 \right]^{1/2} = g \left[ \left( \frac{\delta d}{d} \right)^2 + \left( 2 \frac{\delta T}{T} \right)^2 \right]^{1/2}
\]
You should be careful in the future when using this technique since it only applies to functions of the form \( f = Ax^ny^n \); not all equations in this course will have that simple form.

EVERY MEASURED VALUE IN THIS COURSE SHOULD HAVE AN UNCERTAINTY ASSOCIATED WITH IT. EVERY CALCULATED VALUE IN THIS COURSE SHOULD HAVE A CALCULATED UNCERTAINTY. It is surprising how many students seem to forget this fact between the first and second experiments, resulting in the loss of many points.

**Part 2**

Every time you moved the slider, you changed the mass distribution and effectively created a new pendulum. The question is, which of those pendulums had identical periods when pivoted at both points? When you identify that pendulum, then you can apply equation (8) to it.

Using Excel or a similar program, plot the period (vertical axis) as a function of slider position (horizontal axis) for each of the two pivot points, on the same graph. Have each data set fit to a polynomial of degree 2 and display the equation on your plot. Make sure your axes are appropriately labeled and that there is a legend that makes sense (i.e., one that conveys more information than the default “Series 1”).

Where those two lines cross, you have found a pendulum that produces the same period with two different hanging points. Determine the period at this crossing point, either by reading it off the graph directly or by using the fitting equations to seek the common solution. Since this fitting procedure accounts for trends in several data points, it produces a more accurate value than merely looking for the slider position in your raw data that produces nearly identical periods. The final step in the calculation is straightforward:

Returning to

\[
g = 4\pi^2 \frac{d_1 + d_2}{(T^*)^2},
\]

let’s set \( D = d_1 + d_2 \). There is uncertainty in values for both \( D \) and \( T^* \). The value of and uncertainty in \( D \) has been given by the manufacturer: \( 0.2481 \pm 0.00005 \) m. In this experiment, the uncertainty in \( T^* \) is a bit hard to determine since it is extracted from your plot. You can use your own best judgment on this, but it will probably be somewhere around \( 0.0001 \) s, the precision limit of the electronic timer. Once the uncertainty has been established, then you can perform the remaining analysis of the uncertainty as described in Part 1. The much smaller uncertainty in \( g \) calculated using the Kater pendulum will allow you to display more significant digits in your reported result.
A word on significant digits

Recall that the number of significant digits is the number of digits shown when the number is written in scientific notation. That is, \(0.035=3.5\times10^{-2}\) has two significant digits. Note that in this context that \(0.35\neq0.350\) since the latter expression contains more significant digits and represents a claim to know the value to the nearest thousandth, not merely the nearest hundredth. In general, the number of significant digits you show will depend on the level of uncertainty in your data. There are two principles to remember:

1. We usually show one or maybe two significant digits in the uncertainty calculation, but in this course there will never be any point in showing more than two digits. That is, if you are estimating the number of pennies in a large jar, it may be reasonable to say your count is good to the nearest 100 pennies, or perhaps that your count is good to \(\pm110\) pennies, but it is unreasonable to write \(\pm113\) pennies because if you know your uncertainty to the nearest penny, then you probably could have made a more accurate count in the first place!

2. Make the level of precision in your best value consistent with the uncertainty. If you calculate a force whose uncertainty is \(0.3\) N, then you should show the force to the nearest tenth of a Newton, since that is where the uncertainty lies.

The following are some simple examples:

<table>
<thead>
<tr>
<th>Correct</th>
<th>Incorrect</th>
</tr>
</thead>
<tbody>
<tr>
<td>23±4</td>
<td>23.1±4</td>
</tr>
<tr>
<td>23.2±1.2</td>
<td>23±1.2</td>
</tr>
<tr>
<td>3.50±0.03</td>
<td>3.5±0.03</td>
</tr>
<tr>
<td>3.55±0.12</td>
<td>3.552±0.123</td>
</tr>
</tbody>
</table>

These rules imply that you will need to calculate your uncertainty first before you determine how many significant digits in your best value to use in your tables. If using Excel to set up your tables, remember that it has a tendency to truncate trailing zeros. If these zeroes are significant digits, you need to force Excel to retain them by using a Format:Number process.
What you will turn in for this experiment

All you need to submit for this lab is a partial Results section: You should have tables with your raw data, a properly labeled plot showing the intersection between the two different curves, and your calculations of g from Part 1 and Part 2, including the uncertainty calculation, as outlined above. I need to see the details of the calculations (i.e., what numbers you plugged in where), not just the answers. I refer to this as a “partial Results section” since in a regular report, the Results section will contain text as well as data and calculations. You’ll do that for your next experiment.

Add to the end of your calculation a paragraph discussing the highlights of the uncertainty analysis. In particular, discuss the role of the instrument and of the operator of the instrument in producing both systematic and random error, the way in which data analysis can introduce systematic error (even if you do not make mistakes with the equations!) and to what extent these effects are accounted for in your values of g. Based on your numerical results (not on your procedure and prior expectations) which value of g should you use in your future calculations, 1A, 1B, or 2?

Finally, don’t forget to indicate your lab partner. In future reports, this information will go on the Title/Abstract page.