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Heat kernels for a class of degenerate elliptic operators using stochastic method†

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Formulas for heat kernels are found for degenerate elliptic operators by finding the probability density of the associated Ito diffusion. The formulas involve an integral of a product between a volume function and an exponential term.

Keywords: heat kernel; Brownian motion; degenerate elliptic operator

AMS Subject Classifications: Primary: 53C17; Secondary: 34K10, 35H20

1. Introduction

The goal of this article is to construct heat kernels for a class of degenerate elliptic operators with double characteristics that are hypoelliptic, such as

\[ L_1 = \frac{1}{2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + (x_1^2 + x_2^2) \partial_{x_3}^2 \right), \]

\[ L_2 = \frac{1}{2} \left( \partial_{x_1}^2 + x_1^2 (\partial_{x_2}^2 + \partial_{x_3}^2) \right), \]

\[ L_3 = \frac{1}{2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + x_1^2 \partial_{x_3}^2 \right). \]

The last operator is the well-known example of Baouendi and Goulaouic [1] of degenerate elliptic operator with analytic coefficients, hypoelliptic but not analytic-hypoelliptic.

We emphasize that we do not claim that the formulas presented here are new. They can also be obtained by other methods, such as by using the complex Hamiltonian mechanics or the Fourier transform. Our approach will use stochastic

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†Dedicated to Prof Robert Gilbert on the occasion of his eightieth birthday.
processes and Feynman–Kac’s formula. The main idea here is that the heat kernel is obtained as the probability density of the associated stochastic process. Standard procedures of using expectation operator is used to retrieve the probability density. The reader can find a more detailed approach on this subject [2, Chap. 8.]

The fundamental solutions for the operators (1.1)–(1.3) have been computed in [3] using complex Hamiltonian mechanics. A geometric method has been applied in [4] to find heat kernels. The advantage of using the stochastic method over the geometric method is that it is much shorter. Unlike in the geometric method case, where one needs to solve the bicharacteristics system, compute the action function, and solve the transport equation, the stochastic method uses more powerful formulas like Feynman–Kac’s and calculus involving the expectation operator. Sometimes the probabilistic methods cannot be easily replaced by any of the other known methods, see [5].

Probabilistic methods have been used to find heat kernels since 1930s, when Kolmogorov investigated the heat kernel of the degenerated operator \( \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial y} \) on \( \mathbb{R}^2 \), see [6]. These methods have been applied independently by Hulanicki [7] and Gaveau [8] in late 1970s to find the heat kernel for the Heisenberg–Laplacian on \( \mathbb{R}^3 \) given by
\[
\frac{1}{2} \left( \frac{\partial_x^2}{x} + 2y \frac{\partial_z}{z} \right)^2 + \frac{1}{2} \left( \frac{\partial_y}{y} - 2x \frac{\partial_z}{z} \right)^2.
\]

This article is organized as follows. Section 2 briefly reviews the necessary results to compute heat kernels, such as Feynman–Kac’s formula, Ito diffusions and related issues. Each of the Sections 3–7 deals with a certain degenerate operator and its heat kernel. The formulas are exact and somehow similar; they all involve integrals of products of two terms: a quotient term and an exponential term.

2. Preliminaries

Consider the differential operator
\[
A = \frac{1}{2} \sum_{i,j=1}^{n} \left( \sigma(x) \sigma^T(x) \right)_{ij} \partial_x^i \partial_x^j + \sum_{k=1}^{n} b_k(x) \partial_k,
\]
where \( \sigma(x) \in M_{n \times n} \) matrix function and \( b_k(x) \) smooth functions. The operator \( A \) can be regarded as the generator of the stochastic process \( X_t \) given by the Ito diffusion
\[
dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x,
\]
where \( W_t = (W_1(t), \ldots, W_n(t)) \) is a Brownian motion on \( \mathbb{R}^n \). The relation between the diffusion \( X_t \) and its generator \( A \) is given by
\[
Af(x) = \lim_{t \searrow 0} \frac{E^x[f(X_t)] - f(x)}{t}, \quad f \in C^2_0(\mathbb{R}),
\]
where \( E^x \) is the conditional expectation operator, given \( X_0 = x \), see for instance [9, p. 121].

A result which assures the existence and uniqueness of a diffusion \( X_t \) that is associated with the given operator (2.1) is given by Theorem 5.2.1 of [9].
THEOREM 1 If the time-homogeneous Ito diffusion (2.2) satisfies the Lipschitz condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y| \quad \forall x, y \in \mathbb{R},$$

(2.3)

then there is a unique solution \((X_t)_{t\geq 0}\) with \(X_0 = x\).

The expectation operator \(E = E^x\) is defined as

$$E[f(X_t)] = \int f(y) p_t(x, y) \, dy$$

for any function \(f\) for which the previous integral makes sense. This can be used to retrieve the probability density function as

$$p_t(x_0, x) = \int \delta(y - x) p_t(x_0, y) \, dy = E[\delta(X_t - x)].$$

(2.4)

Since the probability density \(p_t(x_0, x)\) of the stochastic process \(X_t\) is the heat kernel of the operator \(A\) given by (2.1), then formula (2.4) will be used to compute the heat kernel of operators for which the diffusion \(X_t\) is known.

Let \(V\) be a lower-bounded continuous potential function on \(\mathbb{R}^n\). Then the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = Au - Vu, \quad t > 0,$$

$$u(0, x) = f(x)$$

is given by the Feynman–Kac’s formula

$$u(t, x) = E^x[f(X_t) e^{-\int_0^t V(X_s) \, ds}],$$

(2.5)

where the operator \(E^x\) denotes the conditional expectation given that \(X_0 = x\). In the particular case, when \(f\) is a Dirac distribution, we have

$$\frac{\partial p_t}{\partial t} = (A - V) p_t, \quad t > 0,$$

$$\lim_{t \downarrow 0} p_t(x_0, x) = \delta(x - x_0),$$

where \(p_t(x_0, x)\) denotes the heat kernel. Hence Feynman–Kac’s formula provides a formula for the heat kernel of the operator \(A - V(x)\)

$$p_t(x_0, x) = E^x[\delta(X_t - x) e^{-\int_0^t V(X_s) \, ds}].$$

(2.6)

Therefore, it suffices to compute the expectation (2.5). This can be done explicitly just for a few particular cases of the potential \(V(x)\). For instance, in the case of the quadratic potential, it is known that the heat kernel for the Hermite operator which satisfy

$$\frac{\partial p_t}{\partial t} = \frac{1}{2} \sigma^2 p_t - \frac{\eta}{2} x^2 p_t, \quad t > 0,$$

$$\lim_{t \downarrow 0} p_t(x_0, x) = \delta(x - x_0),$$
is given by (see, for instance [2])

\[ p_t(x_0, x) = \frac{1}{\sqrt{2\pi t}} \sqrt{\frac{\eta t}{\sinh(\eta t)}} e^{-\frac{1}{2} \frac{\eta^2}{\sinh(\eta t)}(x^2 + (x_0)^2) \cosh(\eta t) - 2x x_0}. \]

Therefore (2.6) becomes

\[ E[\delta(X(t) - x)e^{-\frac{1}{2} \int_0^t X_s^2 ds}] = \frac{1}{\sqrt{2\pi t}} \sqrt{\frac{\eta t}{\sinh(\eta t)}} e^{-\frac{1}{2} \frac{\eta^2}{\sinh(\eta t)}(x^2 + (x_0)^2) \cosh(\eta t) - 2x x_0}. \tag{2.7} \]

In the case of the linear potential, the heat kernel that satisfies

\[ \frac{\partial p_t}{\partial t} = \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - ax \right) p_t, \quad t > 0, \]

\[ \lim_{t \to 0} p_t(x_0, x) = \delta(x - x_0), \]

is given by the formula (see, for instance, [2, Theorem, 3.15.1])

\[ p_t(x_0, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2} \frac{\eta^2}{\sinh(\eta t)}(x^2 + 2x(x_0)t + \frac{1}{2} \eta^2 t^2)}, \quad t > 0, \]

which can be written as the Feynman–Kac’s formula

\[ E[\delta(X_t - x)e^{-\int_0^t a X_s ds}] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2} \frac{\eta^2}{\sinh(\eta t)}(x^2 + 2x(x_0)t + \frac{1}{2} \eta^2 t^2)}, \tag{2.8} \]

where \( X_t \) is the associated diffusion starting at \( x_0 \) at \( t = 0 \).

In the case of quartic potential, the heat kernel which satisfies

\[ \frac{\partial p_t}{\partial t} = \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - x^4 \right) p_t, \quad t > 0, \]

\[ \lim_{t \to 0} p_t(x_0, x) = \delta(x - x_0) \]

is given by the Feynman–Kac’s formula

\[ p_t(x_0, x) = E[\delta(W_t + x - x_0)e^{-\int_0^t (W_s + x)^4 ds}], \]

which cannot be computed explicitly using elementary functions.

Next, we shall present an identity which will often be used in the calculus with expectations. Consider the stochastic process

\[ Z_t = e^{\int_0^t X_s ds + \frac{1}{2} \int_0^t X_s^2 ds}, \]

where \( X_t \) is a one-dimensional process satisfying \( E[\int_0^T X_t^2 dt] < \infty \), with \( T \) finite, and \( W_t \) denotes a one-dimensional Brownian process starting at 0. If Novikov’s condition \( E[e^{\frac{1}{2} \int_0^t X_s^2 ds}] < \infty \) is satisfied, then \( Z_t \) becomes a martingale, see for instance [9, p. 55].

If the process \( X_t \) is independent of \( W_t \), the martingale condition becomes

\[ E[e^{\int_0^t X_s ds}] = e^{\frac{1}{2} \int_0^t X_s^2 ds} \quad \forall t > 0. \tag{2.9} \]

This identity will be used in the sequel several times.
2. The operator $L_1 = \frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2 + (x_1^2 + x_2^2)\partial_{x_3}^2)$

This is a degenerate elliptic operator which is analytic-hypoelliptic. It is a differential operator of the type given by the Equation (2.1), with the diffusion matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{x_1^2 + x_2^2} \end{pmatrix}, \quad b_k = 0, \quad k = 1, 2, 3.$$ 

One can easily verify that the Lipschitz condition (2.3) holds

$$|\sigma(x) - \sigma(y)| = \left|\sqrt{x_1^2 + x_2^2} - \sqrt{y_1^2 + y_2^2}\right|$$

$$= \frac{|x_1 + y_1|}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} |x_1 - y_1|$$

$$+ \frac{|x_2 + y_2|}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} |x_2 - y_2|$$

$$\leq |x_1 - y_1| + |x_2 - y_2| \leq 2|x - y|,$$

and hence there is a unique solution for the stochastic differential equation (2.2).

In fact, we can solve for the associated Ito diffusion process $X_t = (X_1(t), X_2(t), X_3(t))$ as in the following

$$dX_1(t) = dW_1(t) \Rightarrow X_1(t) = x_1^0 + W_1(t),$$

$$dX_2(t) = dW_2(t) \Rightarrow X_2(t) = x_2^0 + W_2(t),$$

$$dX_3(t) = \sqrt{X_1(t)^2 + X_2(t)^2} \, dW_3(t)$$

$$\Rightarrow X_3(t) = x_3^0 + \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s),$$

where we have the initial condition $(X_1(0), X_2(0), X_3(0)) = (x_1^0, x_2^0, x_3^0)$ and $t \geq 0$.

The two-dimensional process $(X_1(t), X_2(t))$ can be seen as a Brownian motion in the plane that starts at $(x_1^0, x_2^0)$ at time $t = 0$. The distance from the point $(X_1(t), X_2(t))$ to the origin is given by the process $R_t = \sqrt{X_1(t)^2 + X_2(t)^2}$, which is a two-dimensional Bessel process starting at $(x_1^0, x_2^0)$. The process $R_t$ is not smooth at $t = 0$, but since $(W_1(t), W_2(t))$ never hits the origin almost surely, the Ito lemma can still be applied to yield the following stochastic differential equation

$$dR_t = \frac{W_1(t)}{R_t} \, dW_1(t) + \frac{W_2(t)}{R_t} \, dW_2(t) + \frac{1}{2R_t} \, dt, \quad R_0 = 0. \quad (3.1)$$

Therefore, the component $X_3(t)$ given by

$$X_3(t) = x_3^0 + \int_0^t R_s \, dW_3(s)$$

can be interpreted as the work done by the continuous force $R_t$ with respect to the Brownian motion $W_3(t)$. It worth noting that the one-dimensional processes $X_i(t)$
and \( X_2(t) \) are mutually independent, while the process \( X_3(t) \) depends on \( X_1(t) \) and \( X_2(t) \).

The heat kernel of the differential operator \( L_1 \) will be computed using formula (2.4)

\[
p_t(x_0, x) = E[\delta(X_t - x)] = E[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(X_3(t) - x_3)]
= E\left[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(x_0^3 - x_3 + \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s)}\right]
= \frac{1}{2\pi} \int e^{i\eta(x_0^3 - x_3)} E\left[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)e^{i\eta \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s)}\right] \, d\eta,
\]

where we expressed the Dirac distribution as the integral

\[
\delta(y) = \frac{1}{2\pi} \int e^{iy\eta} \, d\eta.
\]

In the following \( E_{w_j} \) will denote the expectation taken with respect to the Brownian motion \( W_j \). Since \( X_1(t) \) and \( X_2(t) \) are mutually independent and independent of \( W_3(t) \), the expectation in (3.2) can be computed using standard properties of expectation operator and applying formula (2.9)

\[
E\left[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(x_0^3 - x_3 + \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s)}\right]
= E_{w_1,w_2}\left[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(x_0^3 - x_3 + \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s)}\right]
= E_{w_1,w_2}\left[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)e^{-\frac{\eta t}{2} \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, ds}\right]
= E_{w_1}\left[\delta(X_1(t) - x_1)e^{-\frac{\eta t}{2} \int_0^t X_1(s)^2 \, ds}\right] E_{w_2}\left[\delta(X_2(t) - x_2)e^{-\frac{\eta t}{2} \int_0^t X_2(s)^2 \, ds}\right].
\]

The previous two expectations have similar expressions and they are given by formula (2.7). For \( j = 1, 2 \) we have

\[
E_{w_j}\left[\delta(X_j(t) - x_j)e^{-\frac{\eta t}{2} \int_0^t X_j(s)^2 \, ds}\right] = \frac{1}{\sqrt{2\pi t}} \frac{e^{-\frac{\eta^2}{4t}}}{\sinh(\eta t)} \frac{\eta t}{\sinh(\eta t)} e^{-\frac{t}{2\sinh(\eta t)}(x_0^2 + x_j^2 \cosh(\eta t) - 2x_0 x_j^2)}.
\]

Substituting the above in (3.3) yields

\[
E[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)e^{i\eta \int_0^t \sqrt{X_1(s)^2 + X_2(s)^2} \, dW_3(s)}]
= \frac{1}{2\pi t \sinh(\eta t)} e^{-\frac{\eta^2}{2t \sinh(\eta t)}(x_0^2 + x_1^2 + x_2^2 \cosh(\eta t) - 2x_0 x_1 x_2)}.
\]

Substituting (3.4) into (3.2) leads to the final expression for the heat kernel of operator \( L_1 \)

\[
p_t(x_0, x) = \frac{1}{(2\pi t)^{3/2}} \int \frac{\eta t}{\sinh(\eta t)} e^{i\eta(x_0^3 - x_3) - \frac{t}{2 \sinh(\eta t)}(x_0^2 + x_1^2 + x_2^2 \cosh(\eta t) - 2x_0 x_1 x_2)} \, d\eta, \quad t > 0.
\]
Consider now the degenerate operator on \( \mathbb{R}_x^k \times \mathbb{R}_y \)

\[
L_{1,k} = \frac{1}{2} \sum_{j=1}^{k} \partial_{x_j}^2 + \frac{1}{2} (x_1^2 + \cdots + x_k^2) \partial_{y_j}^2 = \frac{1}{2} \sum_{j=1}^{k} \partial_{x_j}^2 + \frac{1}{2} |x|^2 \partial_{y_j}^2.
\]

This is the generator of the Ito diffusion

\[
dX_j(t) = dW_j(t) \implies X_j(t) = x_j^0 + W_j(t), \quad j = 1, \ldots, k,
\]

\[
dY(t) = \sqrt{X_1(t)^2 + \cdots + X_k(t)^2} \, dW_{k+1}(t)
\]

\[
\implies Y(t) = y^0 + \int_0^t \sqrt{X_1(s)^2 + \cdots + X_k(s)^2} \, dW_{k+1}(s).
\]

A similar procedure with the one before leads to the following expression for the heat kernel of \( L_{1,k} \)

\[
p_t(x_0, x) = \frac{1}{(2\pi t)^{k+1}} \int \left( \frac{\eta t}{\sinh(\eta t)} \right)^{k/2} e^{i\eta(x^0 - y)} e^{\frac{i}{2} \frac{\eta t}{\sinh(\eta t)} [(|x|^2 + |x_0|^2) \cosh(\eta t) - 2(x, x_0)]} \, d\eta.
\] (3.5)

which after making the substitution \( \tau = \eta t \) becomes

\[
p_t(x_0, x) = \frac{1}{(2\pi t)^{k+1}} \int \left( \frac{\tau}{\sinh(\tau)} \right)^{k/2} e^{i\tau(y^0 - y)} e^{\frac{1}{2} \frac{\tau}{\sinh(\tau)} [(|x|^2 + |x_0|^2) \cosh \tau - 2(x, x_0)]} \, d\tau.
\] (3.6)

It is worth noting that the previous integral cannot be computed explicitly. One can arrive at the same formula using the complex Hamiltonian method, see for instance [4]. In this theory the terms

\[
V(\tau) = \left( \frac{\tau}{\sinh(\tau)} \right)^{k/2},
\]

\[
f(x, y, x_0, y_0; \tau) = -i\tau(y^0 - y) + \frac{1}{2} \frac{\tau}{\sinh(\tau)} [(|x|^2 + |x_0|^2) \cosh \tau - 2(x, x_0)]
\]

have certain geometric significance; the former is the volume function and the latter is the complex modified action function.

In the end of this section we shall make some comments on the heat kernel of the degenerate operator

\[
L^{(m)}_1 = \frac{1}{2} (\partial_{x_1}^2 + \partial_{x_2}^2 + (x_1^{2m} + x_2^{2m}) \partial_{x_3}^2)
\] (3.7)

with \( m \geq 1 \) natural number. The associated diffusion is given by

\[
dX_1(t) = dW_1(t) \implies X_1(t) = x_1^0 + W_1(t),
\]

\[
dX_2(t) = dW_2(t) \implies X_2(t) = x_2^0 + W_2(t),
\]

\[
dX_3(t) = \sqrt{X_1(t)^{2m} + X_2(t)^{2m}} \, dW_3(t)
\]

\[
\implies X_3(t) = x_3^0 + \int_0^t \sqrt{X_1(s)^{2m} + X_2(s)^{2m}} \, dW_3(s).
\]
Using the same method as before we arrive at a formula similar to that of (3.3)
\[ E_{w_1} \left[ \delta(X_1(t) - x_1)e^{-\frac{1}{2}t\int_0^t X_1(s)^2 ds} \right] E_{w_2} \left[ \delta(X_2(t) - x_2)e^{-\frac{1}{2}t\int_0^t X_2(s)^2 ds} \right]. \]

These expectations have closed form expressions for \( m = 1 \). The case \( m = 2 \) for instance, is equivalent with a formula for the heat kernel of \( \frac{1}{2}\partial_x^2 \) with a quartic potential, which is an unsolved yet problem, and perhaps not solvable in terms of elementary functions. Let us suppose, however, that there is a formula for the above expectations

\[ p_t^{(m)}(x_j^0, x_j; \eta) = E_{w_j} \left[ \delta(X_j(t) - x_j)e^{-\frac{1}{2}t\int_0^t X_j(s)^2 ds} \right], \quad j = 1, 2. \]

Then substituting in (3.2) yields the heat kernel for \( L_1^{(m)} \)

\[ p_t(x^0, x) = \frac{1}{2\pi} \int e^{i\eta(X^0 - x)} p_t^{(m)}(x^0_1, x_1; \eta) p_t^{(m)}(x^0_2, x_2; \eta) \, d\eta. \]

4. The operator \( L_2 = \frac{1}{2}(\partial_x^2 + x_1^2(\partial_x^2 + \partial_{x_1}^2)) \)

The operator \( L_2 \) is a Grushin-type operator with two missing directions, which is analytic-hypoelliptic. Its associated diffusion process satisfies

\[
\begin{align*}
\mathrm{d}X_1(t) &= \mathrm{d}W_1(t) \implies X_1(t) = x_1^0 + W_1(t), \\
\mathrm{d}X_2(t) &= X_1(t) \, \mathrm{d}W_2(t) \implies X_2(t) = x_2^0 + \int_0^t X_1(s) \, \mathrm{d}W_2(s), \\
\mathrm{d}X_3(t) &= X_1(t) \, \mathrm{d}W_3(t) \implies X_3(t) = x_3^0 + \int_0^t X_1(s) \, \mathrm{d}W_3(s),
\end{align*}
\]

with \( t \geq 0 \) and \( X_i(0) = x_i^0, \quad i = 1, 2, 3 \). Since the expressions of \( X_2(t) \) and \( X_3(t) \) are similar (having the significance of work made by \( X_1 \) with respect to two independent Brownian motions), the heat kernel should be symmetric in the second and the third arguments.

Using formula (2.4) the heat kernel of the differential operator \( L_2 \) is

\[
\begin{align*}
p_t(x^0, x) &= E[\delta(X_t - x)] = E[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(X_3(t) - x_3)] \\
&= E \left[ \delta(X_1(t) - x_1)\delta(x_0^1 - x_2 + \int_0^t X_1(s) \, \mathrm{d}W_2(s)) \right] \\
&\quad \times \delta(x_0^2 - x_3 + \int_0^t X_1(s) \, \mathrm{d}W_3(s)) \\
&= E \left[ \delta(X_1(t) - x_1) \frac{1}{2\pi} \int e^{i\eta_2(x_0^0 - x_2 + \int_0^t X_1(s) \, \mathrm{d}W_2(s))} \, d\eta_2 \right] \\
&\quad \cdot \frac{1}{2\pi} \int e^{i\eta_3(x_0^0 - x_3 + \int_0^t X_1(s) \, \mathrm{d}W_3(s))} \, d\eta_3.
\end{align*}
\]
The expectation operator in (4.1) is taken with respect to all three Brownian motions $W_i$, $i = 1, 2, 3$. Since $X_1(t)$ is independent of $W_2(t)$ and $W_3(t)$, we have

$$E \left[ \delta(X_1(t) - x_1) \right] e^{i \eta_2 \int_0^t X_1(s) dW_2(s)} e^{i \eta_3 \int_0^t X_1(s) dW_3(s)}$$

Substituting (4.2) into (4.1) yields the following formula for the heat kernel of $L_2$

$$p_t(x^0, x) = \frac{1}{(2\pi)^3} \int \int \sqrt{\eta_2^2 + \eta_3^2 t},$$

where we used Equation (2.9) and Feynman–Kac’s formula for quadratic potential (2.7) to compute the expectations. Substituting (4.2) into (4.1) yields the following formula for the heat kernel of $L_2$

$$L_{2,n} = \partial_{x_1}^2 + x_1^2 \sum_{j=2}^n \partial_{x_j}^2$$

is given by

$$p_t(x^0, x) = \frac{1}{(2\pi)^{n+1}} \int \int \sqrt{\eta_1^2 t},$$

where \( \eta_1 = (\eta_2, \ldots, \eta_n) \), \( \bar{x} = (x_2, \ldots, x_n) \), \( \bar{x}^0 = (x_2^0, \ldots, x_n^0) \), \( |\eta_1| = \sqrt{\eta_2^2 + \cdots + \eta_n^2} \) and \( d\bar{\eta}_1 = d\eta_2 \cdots d\eta_n \).
5. The operator \( L_3 = \frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2 + x_3^2 \partial_{x_3}^2) \)

This is the well-known example given by Baouendi and Goulaouic [1] as a degenerate elliptic operator with analytic coefficients, hypoelliptic but not analytic-hypoelliptic. From the stochastic point of view, \( L_3 \) is the infinitesimal generator of the following Ito diffusion

\[
\begin{align*}
\mathrm{d}X_1(t) &= \mathrm{d}W_1(t) \implies X_1(t) = x_1^0 + W_1(t), \\
\mathrm{d}X_2(t) &= \mathrm{d}W_2(t) \implies X_2(t) = x_2^0 + W_2(t), \\
\mathrm{d}X_3(t) &= X_1(t) \mathrm{d}W_3(t) \implies X_3(t) = x_3^0 + \int_0^t X_1(s) \mathrm{d}W_3(s),
\end{align*}
\]

where \( W_1(t), W_2(t), W_3(t) \) are one-dimensional independent Brownian motions starting at the origin at time \( t = 0 \). We note that the processes \( X_1(t) \) and \( X_2(t) \) are independent of \( W_3(t) \). We can think of \( (X_1(t), X_2(t), X_3(t)) \) as coordinates of the trajectory of a point in \( \mathbb{R}^3 \), which starts at \( (x_1^0, x_2^0, x_3^0) \) at time \( t = 0 \). The first two coordinates are independent Brownian motions, while the third depends on the first coordinate as an integral with respect to \( W_3(t) \). The fact that \( X_2(t) \) is independent of \( X_1(t) \) and \( X_3(t) \) will correspond in the expression of the heat kernel to a multiplicative Gaussian factor. The coordinate \( X_3(t) \) can be interpreted as the work done by \( X_1 \) with respect to the Brownian motion \( W_3(t) \). This can also be thought of as a Brownian motion constrained to move along a non-integrable distribution.

The general theory states that the heat kernel of \( L_3 \) is the probability density \( p_t(x^0, x) \) of the process \( X(t) = (X_1(t), X_2(t), X_3(t)) \); we shall denote \( x = (x_1, x_2, x_3) \) and \( x^0 = (x_1^0, x_2^0, x_3^0) \). Standard formula (2.4) for finding probability density as an expectation yields

\[
p_t(x^0, x) = E[\delta(X_1(t) - x_1)\delta(X_2(t) - x_2)\delta(X_3(t) - x_3)]
\]

\[
= \frac{1}{2\pi} \int e^{\eta(x_1 - x_1)} E \left[ \delta(x_1 - X_1(t)) \delta(x_2 - X_2(t)) e^{i\eta \int_0^t X_1(s) \mathrm{d}W_3(s)} \right] \mathrm{d}\eta,
\]

(5.1)

where we used the representation of the last Dirac distribution as an integral.

Now, the expectation is taken with respect to all processes \( (W_1, W_2, W_3) \). Using the assumption that \( X_1(t) \) and \( X_2(t) \) are mutually independent, and independent of \( W_3 \), we can split the expectation as follows

\[
E \left[ \delta(x_1 - X_1(t)) \delta(x_2 - X_2(t)) e^{i\eta \int_0^t X_1(s) \mathrm{d}W_3(s)} \right]
\]

\[
= E_{W_2} [\delta(x_2 - X_2(t))] E_{W_1} \left[ \delta(x_1 - X_1(t)) E_{W_3} \left[ e^{i\eta \int_0^t X_1(s) \mathrm{d}W_3(s)} \right] \right]
\]

\[
= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x_2 - x_2^0)^2}{2t}} E_{W_1} \left[ \delta(x_1 - X_1(t)) e^{-\frac{i\eta}{2} \int_0^t X_1^2(s) \mathrm{d}s} \right],
\]

(5.2)

where we used (2.9). Applying Feynman–Kac’s formula with quadratic potential (2.7) yields

\[
E_{W_1} \left[ \delta(x_1 - X_1(t)) e^{-\frac{i\eta}{2} \int_0^t X_1^2(s) \mathrm{d}s} \right] = \frac{1}{\sqrt{2\pi t}} \int \frac{\eta t}{\sinh(\eta t)} e^{-\frac{\eta t}{2} \sinh^2(x_1^0 + x_2^0 t)} \cosh(\eta t) - 2x_1 x_2^0 \mathrm{d}x_1.
\]
Substituting successively in (5.2) and (5.1) yields the following expression for the heat kernel
\[ p_t(x^0, x) = \frac{1}{(2\pi)^3} e^{-\frac{(x^2 - x_0^2)^2}{2t} \int \frac{\eta I}{\sinh(\eta \tau)} e^{\eta (\xi_0 - x_0)} - \frac{\eta}{2} \frac{\partial^2}{\partial \xi_0^2} \cosh(\eta \tau) - 2\xi_0 x_0} d\eta, \] for \( t > 0 \). We note that this is the product between a Gaussian in the \( x_2 \) variable and the heat kernel of the two-dimensional Grushin operator, fact which does not surprise us, since the operator \( L_3 \) is the sum of two commuting operators (the Grushin operator \( \frac{1}{2}(\partial_{x_1}^2 + x_1^2 \partial_{x_3}^2) \) and \( \frac{1}{2} \partial_{x_2}^2 \)).

6. The operator \( L_{kp} = \frac{1}{2}(\Delta_x + |x|^2 \Delta_y) \)

In this section we shall find an exact formula for the heat kernel for the family of degenerate elliptic operators
\[ L_{kp} = \frac{1}{2}(\Delta_x + |x|^2 \Delta_y) = \frac{1}{2} \sum_{j=1}^k \partial_{x_j}^2 + \frac{1}{2} \sum_{j=1}^p (x_j^2 \partial_{y_j}^2 + \partial_{y_j}^2). \] This is a generalization of several of the previous operators
\[ L_{11} = \frac{1}{2} (\partial_{x_1}^2 + x_1^2 \partial_{x_3}^2), \]
\[ L_{12} = L_2 = \frac{1}{2} (\partial_{x_1}^2 + x_1^2 (\partial_{x_3}^2 + \partial_{x_2}^2)), \]
\[ L_{21} = L_1 = \frac{1}{2} (\partial_{x_2}^2 + \partial_{x_3}^2 + (x_1^2 + x_2^2) \partial_{x_3}^2). \]

The Green’s functions of the operators (6.1) can be found in [10].

To clarify notations, set \( x = (x_1, \ldots, x_k), x^0 = (x_1^0, \ldots, x_k^0), y = (y_1, \ldots, y_p) \) and \( y^0 = (y_1^0, \ldots, y_p^0) \). Since these operators are invariant under translations along \( y \)-axis, \( j = 1, \ldots, p \), we may assume that \( y^0 = 0 \). Let \( W_i = (W_1(t), \ldots, W_k(t)) \) and \( B_t = (B_1(t), \ldots, B_p(t)) \) be two Brownian motions on \( \mathbb{R}^k \) and \( \mathbb{R}^p \), respectively. Consider the processes
\[ X_t = (X_1(t), \ldots, X_k(t)), \quad Y_t = (Y_1(t), \ldots, Y_p(t)). \]

Then the Ito diffusion \((X_t, Y_t)\) on \( \mathbb{R}^k \times \mathbb{R}^p \) associated with the operator \( L_{kp} \) satisfies the following stochastic differential equation
\[ dX_t = dW_t, \quad dY_t = |X_t| dB_t. \]

Using the notations \( \eta = (\eta_1, \ldots, \eta_p), \tau = (\tau_1, \ldots, \tau_p), d\tau = d\eta_1 \cdots d\eta_p, d\tau = d\tau_1 \cdots d\tau_p \) and keeping in mind that
\[ \delta(X_t - x) = \delta(X_1(t) - x_1) \cdots \delta(X_k(t) - x_k), \]
\[ \delta(Y_t - y) = \delta(Y_1(t) - y_1) \cdots \delta(Y_p(t) - y_p), \]
and denoting by $E_{W_B}$ the expectation with respect to both Brownian motions $W_t$ and $B_t$, the heat kernel of $L_{kp}$ becomes

$$p_t(x^0, x, y) = E_{W_B} \left[ \delta(X_t - x) \delta(Y_t - y) \right] = E_{w} \left[ \delta(X_t - x) \delta(y + \int_0^t |X_s| \, dB_s) \right]$$

$$= E_{w_B} \left[ \delta(X_t - x) \frac{1}{(2\pi\rho^2)^{d/2}} \int e^{-\rho \sum \eta_j y_j} e^{\int_0^t |X_s| \, dB_s} \right]$$

$$= \frac{1}{(2\pi\rho)^d} \int e^{-\rho \sum \eta_j y_j} E_{w_B} \left[ \delta(X_t - x) e^{\int_0^t |X_s| \, dB_s} \right] \, d\eta$$

$$= \frac{1}{(2\pi\rho)^d} \int e^{-\rho \sum \eta_j y_j} E_{w} \left[ \delta(X_t - x) e^{\int_0^t |X_s|^2 \, ds} \right] \, d\eta$$

$$= \frac{1}{(2\pi\rho)^d} \int e^{-\rho \sum \eta_j y_j} \prod_{m=1}^{k} E_{w_m} \left[ \delta(X_m(t) - x_m) e^{-\frac{1}{2} \sum \eta_j x_m^j \sum \eta_j x_m^j \, ds} \right] \, d\eta$$

$$= \frac{1}{(2\pi\rho)^{d/2}} \int \left( \frac{|\eta|}{\sinh(|\eta|)} \right)^{k/2} e^{-i \eta \cdot y - \frac{1}{2} \sum \eta_j x_m^j \cosh(|\eta|) - 2 \sum \eta_j x_m^j} \, d\eta$$

$$= \frac{1}{(2\pi\rho)^{d/2}} \int \left( \frac{\tau}{\sinh(|\tau|)} \right)^{k/2} e^{-\frac{i}{2} \tau \cdot y - \frac{1}{2} \sum \tau_j x_m^j \cosh(|\tau|) - 2 \sum \tau_j x_m^j} \, d\tau.$$ 

As we mentioned before, when $k = p = 1$, the operator $L_{11}$ is the famous Grushin operator [11]:

$$L_{11} = \partial_x^2 + x^2 \partial_{y_j}^2.$$ 

Using the result for the operator $L_{kp}$, we can derive the heat kernel for the operator $L_{11}$ as follows:

$$p_t(x^0, x, y) = \frac{1}{(2\pi\rho)^{d/2}} \int \left( \frac{\tau}{\sinh(|\tau|)} \right)^{1/2} e^{-\frac{i}{2} \tau \cdot y - \frac{1}{2} \sum \tau_j x_m^j \cosh(|\tau|) - 2 \sum \tau_j x_m^j} \, d\tau.$$ 

For $j, k \in \mathbb{N}$ with $j \neq k$, denote

$$L_{A_j} = \partial_{x_j}^2 + x_j^2 \partial_{y_j}^2 \quad \text{and} \quad L_{A_k} = \partial_{x_k}^2 + x_k^2 \partial_{y_k}^2.$$ 

It is easy to see that $[L_{A_4, L_{A_j}}] = 0$. Hence, the heat kernel for the operator $L_{A_4} + L_{A_j}$ is the product of the heat kernel for the operators $L_{A_j}$ and $L_{A_k}$. Now we have the following corollary.

**Corollary 1** The heat kernel for the operator

$$L_4 = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j^2 \partial_{y_j}^2.$$
\[ p_t(x^0, x, y) = \frac{1}{(2\pi t)^{\frac{n}{2}}} \prod_{j=1}^{n} \int \left( \frac{\tau_j}{\sinh(\tau_j)} \right)^{1/2} e^{-\frac{1}{4} \tau_j y_j - \frac{1}{4} \sum_{j=1}^{n} \left[ (y_j^2 + \xi_j^2) \cosh(\tau_j) - 2y_j x_j \right]} d\tau_j. \]

It is worth noting that if we apply the previous method for operators of the type

\[ L_{kpm} = \frac{1}{2} (\Delta_x + |x|^2 m \Delta_x), \]

with \( m \geq 1 \) natural, we get into the same computational limitation issues as for the operator (3.7). Only the case \( m = 1 \) can be computed explicitly, while the other cases cannot be computed. This is a limitation of the method; however, no other known method can provide exact solutions for the case \( m \geq 2 \). The advantage of finding the heat kernel as a probability density of the associated diffusion is that is shorter, and hence more efficient than other methods.

7. The operator \( T = \frac{1}{2} \left( \partial_x^2 + x \partial_y^2 \right) \)

This is the well-known operator of Tricomi. We shall consider \( x > 0 \), so the associated diffusion satisfies the equation

\[
\begin{align*}
\frac{dX_t}{dt} &= dW_1(t) \implies X_t = x^0 + W_1(t), \\
\frac{dY_t}{dt} &= \sqrt{X_t} dW_2(t) \implies Y_t = y^0 + \int_0^t \sqrt{X_s} dW_2(s),
\end{align*}
\]

where \( W_1(t) \) and \( W_2(t) \) are two independent one-dimensional Brownian motions. The procedure for finding the heat kernel is similar to the previous cases, the only difference being in using formula (2.8) instead of (2.7), with \( \alpha = \eta^2/2 \)

\[
p_t(x^0, y^0, x, y) = E[\delta(X_t - x)\delta(Y_t - y)] \\
= E \left[ \delta(X_t - x) \delta \left( y^0 - y + \int_0^t \sqrt{X_s} dW_2(s) \right) \right] \\
= E \left[ \delta(X_t - x) \frac{1}{2\pi} \int e^{i\eta(x^0 - y) + \int_0^t \sqrt{X_s} dW_2(s)} d\eta \right] \\
= \frac{1}{2\pi} \int e^{i\eta(x^0 - y)} E \left[ \delta(X_t - x) e^{i\eta \int_0^t \sqrt{X_s} dW_2(s)} \right] d\eta \\
= \frac{1}{2\pi} \int e^{i\eta(x^0 - y)} E_{w_1} \left[ \delta(X_t - x) E_{w_2} \left[ e^{i\eta \int_0^t \sqrt{X_s} dW_2(s)} \right] \right] d\eta \\
= \frac{1}{2\pi} \int e^{i\eta(x^0 - y)} E_{w_1} \left[ \delta(X_t - x) e^{-\frac{1}{2} \eta^2 \int_0^t X_s ds} \right] d\eta \\
= \frac{1}{2\pi} \int e^{i\eta(x^0 - y)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x - x^0)^2}{2\eta^2} + \frac{\eta^2 (x + x^0)^2}{4} + \frac{X_t}{4\pi \eta t}} d\eta \\
= \frac{1}{(2\pi)^{3/2} t^{1/2}} e^{-\frac{(x - x^0)^2}{2\eta^2} - \frac{\eta^2 (x + x^0)^2}{4} + \frac{X_t}{4\pi \eta t}} d\eta.
\]
It worth noting that one can also arrive at the same result if using the partial Fourier transform with respect to $y$.

8. Conclusions

All the operators investigated in this article are on $\mathbb{R}^n$, and the associated Ito diffusions are also on $\mathbb{R}^n$. One of the future direction of investigation is to compute heat kernels of operators on curved surfaces using diffusion on surfaces.

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