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| Abstract | This paper investigates several strategies for prey and predator in both bounded and unbounded domains, assuming they have the same speed. The work describes how the prey should move to escape from the predator and how predator should move to catch the prey. The approach is agent-based and explicitly tracks movement of individuals as prey and predator. We show that the prey escapes one or two competing predators, while might be caught in the case of three predators. The paper also describes a strategy for finding a well camouflaged static prey which emits signals. |

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Geometric optimization for prey–predator strategies

Bader Alshamary · Ovidiu Calin

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Abstract This paper investigates several strategies for prey and predator in both bounded and unbounded domains, assuming they have the same speed. The work describes how the prey should move to escape from the predator and how predator should move to catch the prey. The approach is agent-based and explicitly tracks movement of individuals as prey and predator. We show that the prey escapes one or two competing predators, while might be caught in the case of three predators. The paper also describes a strategy for finding a well camouflaged static prey which emits signals.

Keywords Prey–predator strategy · Regions of influence · Dynamic strategy

0 Introduction

The goal of this paper is to explicitly construct geometrical models that explain several strategies for the prey–predator interaction. The motivation of this work is to provide a mathematically explicit approach of a problem in a field where usually the work is done by performing computer simulations. Here we work the strategy of how the prey should move to escape from the predator and how predator should move to catch the prey without the aid of a computer simulation. The approach is agent-based that explicitly tracks movement of individuals as prey and predator. We also treat the...
interaction between one prey and several predators reducing it to a min–max optimization problem.

The original prey–predator pursuit problem was introduced by Benda et al. (1985) in a grid-world version, where the prey and predator agents are moving one step per the unit of time, either horizontally or vertically. This strategy was improved by Korf (1994), where each predator chooses a step that brings it nearest to the prey. The prey–predator strategies inspired some authors to work on adaptive systems (Holland 1975) and eventually led to genetic algorithms that can develop solutions to optimization problems.

In the case of one prey and several predators, it is important to distinguish whether the predators jointly decide on a strategy or they are just competitors moving individually. Korf (1994) showed that even in the later case the predators can capture the prey if they choose locally optimal strategies. However, the prey–predator pursuit problem is not a completely solved problem for any domain. Some of these problems have been approached algorithmically, see for instance Haynes and Sen (1997). The case of collective strategy of prey by flocking is treated by Nishimura and Ikegami (1996).

The present paper deals with strategies of geometric nature of a prey and one or several predators. The speed of both the prey and its predators are considered equal, unless otherwise stated.

The paper is organized as follows. The first section deals with the strategy of a predator pursuing a prey and shows that if the domain is a disk, the predator can get as close as possible to the prey, while if the domain is unbounded, the prey can keep its predator at a constant distance. The second section describes different strategies of a prey followed by two predators and discusses the strategy at boundary points. The next section deals with the case of one prey and several predators which move discretely by taking jumps of given length in any direction, and describes the case when the prey can be captured by its predators. Section 6 describes a strategy of finding a prey which is well camouflaged and which emits signals that can be detected by the predator. The final section is dedicated to a detailed discussion regarding previous sections.

1 Prey with one predator

In this section we shall discuss the strategy of a predator continuously pursuing a prey and develop an optimal strategy for the prey.

The predator’s strategy Let \( x(t) \) and \( y(t) \) be smooth curves in the interior of a disk \( D \) of radius \( r \), describing the trajectories of the prey and its predator, respectively. The parameter \( t \) denotes the time.

Given the curve \( x(t) \), the predator needs a strategy which decreases the distance \( \rho(t) = |y(t) - x(t)| \) between the prey and the predator. In this paper the predator’s strategy is to have the direction pointing always to the prey’s position, i.e. \( y(t) \) satisfies the equation

\[
y(t) + c y'(t) \rho(t) = x(t), \tag{1.1}
\]
where \( c > 0 \) is a scaling constant. This is an important and critical assumption, which is reasonable if the predators can respond immediately to the prey’s movement. In reality, it might be a time lag for the predators to adjust their position. A discussion of how a slight time lag would change the results will be discussed in the final discussion section.

Another important condition is to assume that the prey and the predator move at the same speed. Without losing the generality, we may also assume that both curves are unit-speed \(|y'(t)| = |x'(t)| = 1\). This implies that the constant \( c \) in relation (1.1) can be scaled to be 1. The other cases regarding different relations between running speeds is treated in the discussion section.

Given the predator’s strategy (1.1) the prey needs to develop a strategy to escape the predator. Differentiating in (1.1) yields

\[
y'(t)(1 + \rho'(t)) + y''(t)\rho(t) = x'(t).
\]

Since \(|y'(t)| = 1\), differentiating with respect to \( t \) in \(|y'(t), y'(t)| = 1|\) and using the symmetry of the inner product yields \(|y''(t), y'(t)| = 0\), i.e. the acceleration is normal to the velocity. This result is true only if the speed is constant. Using \(|x'(t)| = 1, \langle x', y'' \rangle = 0\), and the expression of the curvature \( \kappa = |y''| \), taking the square in (1.2) leads to

\[
(1 + \rho')^2 + \kappa^2 \rho^2 = 1.
\]

Hence \( \kappa < \frac{1}{\rho'} \), i.e. the radius of curvature of the predator’s trajectory \( y(t) \), which is \( \frac{1}{\rho} \), is larger than the distance \( \rho \) from the prey to the predator.

The strategy of the prey Taking the inner product in (1.2) with respect to \( y' \) yields \( \langle x', y' \rangle = 1 + \rho' \). If let \( \theta(t) \) be the angle between the directions \( x'(t) \) and \( y'(t) \), then

\[
\rho' = \cos \theta - 1 \leq 0,
\]

see Fig. 1. Hence the distance between the prey and the predator cannot increase. It stays constant if \( \theta = 0 \), and decreases otherwise.

The optimal strategy for the prey is to keep the distance \( \rho(t) \) constant, by choosing its direction collinear with the predator’s direction. This implies that the prey’s optimal trajectory must be a straight line. From (1.1) it follows that the predator’s trajectory will also be a straight line. Since the domain is bounded, sooner or later the prey will reach the edge of the domain at the point \( P \), see Fig. 2. The aforementioned strategy cannot be applied at border points. The prey should continue its path such that the change in the direction angle is minimum. The tangent to the domain at \( P \) makes with the direction of the prey two angles \( \varphi_1 \), and \( \varphi_2 \). If the direction of the prey is normal to the boundary, the angles are equal to \( \pi/2 \) and the prey chooses to go along the boundary either to the left or to the right. If the direction is slant, the prey can do it best by running along the circumference in the direction of the smallest angle. For instance, in the situation described by Fig. 2 the prey should run along the boundary in the direction of \( A' \) since \( \varphi_1 < \varphi_2 \), which implies \(|CA'| > |CA|\).
The prey’s strategy is to keep the angle $\theta(t)$ as small as possible.

The prey can either go in the direction of $A$ or $A'$. Since $|CA| < |CA'|$ for $|PA| = |PA'|$, it should choose to run in the direction of $A'$ in order to maximize the distance from the predator.

**Proposition 1.1** Assume the prey and the predator can move inside a circular domain. Applying the previous strategy, the predator can get as close as possible to the prey, i.e. $\forall \epsilon > 0, \exists t > 0$ such that $\rho(t) < \epsilon$.

**Proof** Assume the negation of the conclusion holds true

$\exists \epsilon > 0, \forall t > 0, \rho(t) > \epsilon$. \hfill (1.4)

If the prey and the predator are moving inside the disk, the aforementioned optimal strategies imply that they move along a line, and the distance between them is constant. When the prey runs along the boundary, the predator’s trajectory is a curve with the curvature $\kappa$ greater than the curvature of the boundary circle, that is $1/r$, so by using (1.4) we have

$$(1 + \rho')^2 = 1 - \kappa^2 \rho^2 < 1 - \frac{\rho^2}{r^2} < 1 - \frac{\epsilon^2}{r^2}.$$  

Then

$$-1 - \sqrt{1 - \frac{\epsilon^2}{r^2}} < \rho' < -1 + \sqrt{1 - \frac{\epsilon^2}{r^2}} < 0.$$
Let $m_\pm = -1 \pm \sqrt{1 - \epsilon^2 / r^2}$, where $m_- < -1 < m_+ < 0$. Integrating yields

$$\rho_0 + m_- t < \rho(t) < \rho_0 + m_+ t.$$  

This contradicts the assumption $\rho(t) > \epsilon$, since the expression on the right side can be made as negative as we wish by choosing a large enough $t$. Hence (1.4) is false, and the proof is finished. $\square$

The previous result states that the predator captures the prey in an infinite time, provided the prey and the predator have negligible dimensions. However, in real life the predator practically captures the prey when the distance $\epsilon$ becomes smaller than the dimension of the prey.

2 Prey with two predators

We shall treat next the case of two predators pursuing a prey in the interior of a disk of radius $r$, all agents having the same speed. The prey tries to escape the predators, or, in the best case, to maximize its life span. There are two cases to distinguish from:

- the case of competing predators that do not cooperate in their strategies in capturing the prey
- the case of collaborative predators, when the predators do have a collective strategy.

The prey and the predators’ trajectories are denoted by $x(t)$, $y_1(t)$ and $y_2(t)$, respectively. Consider the distances from the prey to each of the predators be

$$\rho_1(t) = |y_1(t) - x(t)|, \quad \rho_2(t) = |y_2(t) - x(t)|.$$  

Consider the optimization problem with the objective function that needs to be maximized given by

$$T = \{t > 0; y_1(t) = x(t) \lor y_2(t) = x(t)\} = \{t > 0; \rho_1(t) = 0 \lor \rho_2(t) = 0\}.$$  

The prey needs to find a strategy to maximize $T$, eventually making $T$ infinite. We shall discuss in the following several strategies of the prey of metrical, topological, or vectorial nature, as well as their relationship. The following strategies may be applied by the prey as long as it does not reach the boundary of the domain.

The maximizing distance strategy This metrical flavored strategy relies on the fact that the prey feels the most danger from the closest predator. It applies when there is a relatively short distance between the prey and its predators. Feeling the most fear from the closest predator, the prey will try to increase the distance from it. Unfortunately, as it was described in the previous section, the best the prey can do is to keep a constant distance from the closest predator. Assume $\rho_1(t_0) < \rho_2(t_0)$, at some time $t_0$. Then the prey feels at time $t_0$ more danger from the first predator and it will try to increase
The prey strategy is to keep equal angles $\theta_1(t) = \theta_2(t)$

the distance $\rho_1$, even in the disfavor of $\rho_2$, see Fig. 3a. This strategy will lead after
some time to equal distances $\rho_1 = \rho_2$. This means the prey moves along a line passing
through the midpoint of the segment $y_1(t)y_2(t)$ and perpendicular on it, see Fig. 3b.
This is known in geometry under the name of perpendicular bisector of a segment.

Let $\theta_1(t), \theta_2(t)$ be the angles made by the vector $\mathbf{x}'(t)$ with $y_1'(t)$ and $y_2'(t)$ respectively. Since the strategy of non-collaborative predators is that both follow the direction
of the prey, we have

\[
\begin{align*}
\mathbf{y}_1(t) + \mathbf{y}_1'(t)\rho_1(t) &= \mathbf{x}(t) \implies \mathbf{y}_1'(t)(1 + \rho_1'(t)) + \mathbf{y}_1''\rho_1(t) = \mathbf{x}'(t) \\
\mathbf{y}_2(t) + \mathbf{y}_2'(t)\rho_2(t) &= \mathbf{x}(t) \implies \mathbf{y}_2'(t)(1 + \rho_2'(t)) + \mathbf{y}_2''\rho_2(t) = \mathbf{x}'(t),
\end{align*}
\]

and using that $|\mathbf{x}'(t)| = |\mathbf{y}_1'(t)| = |\mathbf{y}_2'(t)| = 1$, we get

\[
\begin{align*}
1 + \rho_1'(t) &= \langle \mathbf{x}'(t), \mathbf{y}_1'(t) \rangle \implies \rho_1'(t) = \cos \theta_1(t) - 1 \leq 0 \quad (2.5) \\
1 + \rho_2'(t) &= \langle \mathbf{x}'(t), \mathbf{y}_2'(t) \rangle \implies \rho_2'(t) = \cos \theta_2(t) - 1 \leq 0. \quad (2.6)
\end{align*}
\]

If $\rho_1(t_0) < \rho_2(t_0)$, the prey runs keeping $\theta_1(t) = 0$, i.e. the distance from the
closest predator $\rho_1(t) = \rho_1(t_0)$ constant. Since $\theta_2 > 0$, the distance from the farthest
predator $\rho_2(t)$ decreases until $\rho_2(t_1) = \rho_1(t_1) = \rho(t_0)$. From this moment the prey
will run with $\theta_1(t) = \theta_2(t)$, its trajectory being a straight line. This strategy continues
until the prey reaches the boundary of the domain.

The plane separation strategy This strategy is of a topological nature. It applies when
the prey does not have any metrical knowledge, as in the case when the distances
are either too large with respect to the agents size, or the prey and its predators are
running on a space where a distance function is not defined (such that in the case of
euglena-type agents in bubble water).

The direction line $d(t)$ of first predator’s velocity $\mathbf{y}_1'(t)$ divides the disk at any
time $t$ into two convex regions $H_1(t)$ and $H_2(t)$, see Fig. 4a. Because of the
predator’s strategy (1.1), the prey $x(t)$ belongs to the line $d(t)$. Assume the second predator $y_2(t) \notin d(t)$. Then either $y_2(t) \in H_1(t)$ or $y_2(t) \in H_2(t)$. Let’s say that $y_2(t) \in H_2(t)$. Then the prey strategy is to move into the “safe” region $H_1(t)$ at any time $t$, and never return into an “unsafe” region $H_2(t')$ with $t' < t$. This means that for an infinitesimal change in time $\delta t > 0$ we have

$$x(t + \delta t) = x(t) + x'(t)\delta t \in H_1(t) \quad \text{and} \quad x(t + \delta t) \notin H_2(t') \quad \forall t' \leq t.$$ 

This means that at time $t$ the prey tends to run into the convex set

$$x(t + \delta t) \in C_t = \bigcap_{t' \geq t} H_1(t'),$$

and eventually, run out of room, since the convex sets are descending, i.e., $C_t \subset C_u$ for $t > u$.

**The vectorial strategy** Under this strategy the prey’s direction $x'(t)$ at any time $t$ is given by the direction of the vectorial sum of the velocities of predators $y_1'(t) + y_2'(t)$, i.e.,

$$x'(t) = \frac{y_1'(t) + y_2'(t)}{|y_1'(t) + y_2'(t)|}.$$

The velocities add by the parallelogram rule, and since $|y_1'(t)| = |y_2'(t)|$, the parallelogram becomes a rhombus. Since diagonals in a rhombus are also angle bisectors, it follows that $\theta_1(t) = \theta_2(t)$, where $\theta_i(t)$ is the angle formed by $x'(t)$ with $y_i'(t)$, see Fig. 4b. Since the predators’ strategy is to follow the prey’s direction, using (2.5)–(2.6) it follows that

$$\rho_1'(t) - \rho_2'(t) = 0,$$
Fig. 5  Strategy at a border point: a The oblique case $\phi_r < \phi_l$; b Possible directions of the prey in the case $\phi_r = \phi_l = \frac{\pi}{2}$.

The strategy at border points  All of the previous strategies work well in the case of unbounded domains. For bounded domains they break down at the border points. We shall deal with this case in the following.

Assume the prey reaches the boundary of the domain $D$ at time $t_b$, i.e. $x(t_b) \in \partial D$. The tangent line to $\partial D$ at $x(t_b)$ makes with the prey’s direction $x'(t_b)$ the angles $\phi_l$, $\phi_r$ towards the left and right, respectively. We have two main cases:

(i) The prey’s direction is oblique to the border. In this case the prey will start running along the boundary $\partial D$ in the direction of the smallest angle, since otherwise it will get even closer to one of the predators. The case $\phi_r < \phi_l$ is represented in Fig. 5a.

(ii) The prey’s direction is perpendicular to the border. In this case $\phi_r = \phi_l = \frac{\pi}{2}$ and the prey can go either along the border $\partial D$ to the right or to the left (in the case $d_1(t) > d_2(t)$), or bounce back (if $d_1(t) < d_2(t)$), see Fig. 5b.

3 The strategy of jumping prey and predators

In this section the prey and predator are considered to move discretely in any direction, by taking jumps at the same rate. Their maximum jump distance is denoted by $\epsilon$. The prey and the predator can take alternative jumps. This models, for instance, a bird trying to capture a frog.

Let $x$ and $y$ be the positions at a certain time of the prey and predator, respectively. Then after the next jump the prey and predator can be anywhere inside or on the circles $S(x, \epsilon)$ and $S(y, \epsilon)$, respectively. These domains will be regarded as regions of influence. It is important to note that the predator captures the prey whenever the prey enters its region of influence.
Fig. 6 Min–max strategy: a Optimum position for the prey at point $A$ (maximizing the distance from the predator); b Optimum position for the predator at point $B$ (minimizing the distance to the prey); c Both prey and predator optima positions (maximizing distance for the prey and minimizing distance for the predator)

For instance, the predator can be a hunter that shoots any prey situated at a distance smaller or equal to $\epsilon$. Hence, it is obvious that the prey has to stay as far away as possible from the predators’ region of influence. Next we shall develop a strategy based on the aforementioned observations.

**Min–Max strategy** Any position of the prey outside the predator’s region of influence, the set $S(y, \epsilon)$, is a safe position. However, the prey will try to optimize its option by taking the most distant position from the predator. This will be the “safest position”, i.e. the position where the prey will feel the safest. This position is always realized for a point on the circle $S(x, \epsilon)$, which is collinear with the centers $y$ and $x$. In Fig. 6a the safest position of the prey is situated at the point $A$.

On the other side, the predator’s position should realize the minimum distance from the prey’s region of influence. This is realized for the point $B$, which belongs to the boundary of the predator’s region of influence, see Fig. 6b.

If both the prey and predator apply the aforementioned optimal min–max strategy, then the prey jumps from $x$ to $A$ and the predator jumps from $y$ to $B$, see Fig. 6c. This is consistent with the fact that the predator’s direction points always towards the prey, and the optimal trajectories of the prey and predator are straight lines. We had assumed that the previous jumps are possible, i.e., the strategy is applied away from the boundary of the domain.

### 4 Prey–predator adjusted strategy

In this section we shall assume that the strategies applied by the prey and the predator are dynamic, in the sense that for each jump of the prey, the predator’s response
should be the optimal one. On the other side, for each move of the predator, the prey should adopt an optimal strategy too. We shall present next the optimal strategy when a change in the direction is done by one of the participants. In the following the distance between the prey and predator is denoted by $d = |yx|$.

The next result deals with the optimal response move of the predator, which tries to minimize the distance to the prey, see Fig. 7a. The prey jumps first and the predator jumps after trying to minimize distance.

**Proposition 4.1** Let $A$ and $B$ be the optimum positions for the prey and predator as described by Fig. 6c. Assume that the prey jumps first from $x$ to $A'$, making an angle $\widehat{A'xA} = \varphi \in (-\pi/2, \pi/2)$. Then the optimal move for the predator is to jump from $y$ to $B'$ making an angle $\theta = \widehat{B'yB}$, in the same direction, given by

$$
\sin \theta = \frac{\sin \varphi}{\sqrt{1 + \frac{d^2}{\epsilon^2} + 2\frac{d}{\epsilon} \cos \varphi}}. \quad (4.7)
$$

**Proof** The law of sines applied in the triangle $A'yx$ yields

$$
\sin \theta = \frac{\epsilon \sin \varphi}{|A'y|}. \quad (4.8)
$$
Expressing $|A'y|$ by the law of cosines in triangle $A'yx$

$$|A'y|^2 = \epsilon^2 + d^2 + 2\epsilon d \cos \varphi$$

and substituting in (4.8) leads to the desired result (4.7). □

An iterative application of the the strategy described by Proposition 4.1 leads to a convergent process. The prey and the predator take alternative jumps, the prey being first. More precisely, we have the following estimation result:

**Proposition 4.2** Let $d_n$ be the prey–predator distance after $n$ steps of applying the strategy of Proposition 4.1. Then

$$d_n < d \left( 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d^2}} \right)^n,$$

where $d = d_0$ is the initial distance. In particular, the series $\sum_n d_n$ converges, and $\lim_{n \to \infty} d_n = 0$.

**Proof** Let $d = |yx|$ and $d_1 = |A'B'|$. Writing the area of triangle $yxA'$ in two different ways we have

$$\sigma(yxA') = \frac{1}{2} d \epsilon \sin \varphi$$
$$\sigma(yxA') = \sigma(yxB') + \sigma(B'xA') = \frac{1}{2} d \epsilon \sin \theta + \frac{1}{2} \epsilon d_1 \sin \alpha.$$  

Using $\varphi = \alpha + \theta$, equating the previous two equations yields

$$\frac{d_1}{d} = \frac{\sin \varphi - \sin \theta}{\sin(\varphi - \theta)}.$$  

Substituting $\sin \theta$ from (4.7), and using that $\varphi, \theta, \varphi - \theta \in (0, \frac{\pi}{2})$, leads to the following estimations

$$\frac{d_1}{d} = \frac{\sin \varphi}{\sin(\varphi - \theta)} \left(1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d^2 + 2\epsilon d \cos \varphi}}\right) < 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d^2 + 2\epsilon d \cos \varphi}}$$

The upper bound of the quotient $d_1/d$ does not depend on the angle $\varphi$. It also follows from the previous inequality that $d_1 < d$.

In a similar way, after $n$ steps, we have the inequalities $d_n < d_{n-1} < \cdots < d_1 < d_0 = d$ and

$$\frac{d_n}{d_{n-1}} < 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d_{n-1}^2}} < 1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d_0^2}} \implies d_n < d_{n-1} \left(1 - \frac{\epsilon}{\sqrt{\epsilon^2 + d_0^2}}\right).$$
Iterating yields the desired inequality (4.9). The second part follows from the properties of the geometric series.

The following result is concerned with the optimal response move of the prey, which tries to maximize the distance from its predator, which jumps first, see Fig. 7b.

**Proposition 4.3** Assume the predator jumps first from y to B', with \( \theta \in (-\pi/2, \pi/2) \). Then the optimal response of the prey is to jump from x to A', with A' and B' separated by \( \mathbf{yx} \), such that \( \hat{\mathbf{AxA'}} = \varphi \), with

\[
\sin \varphi = \frac{\epsilon \sin \theta}{\sqrt{\epsilon^2 + d^2 - 2\epsilon d \cos \theta}}.
\]

**Proof** The law of sines and the cosine theorem applied in triangle \( \mathbf{yxB'} \) yields

\[
\frac{\epsilon}{\sin \varphi} = \frac{|B'\mathbf{x}|}{\sin \theta} \quad \text{(4.11)}
\]

\[
|B'\mathbf{x}|^2 = d^2 + \epsilon^2 - 2\epsilon d \cos \theta. \quad \text{(4.12)}
\]

Eliminating \( |B'\mathbf{x}| \) from the previous expressions leads to (4.10).

**5 Capturing the prey**

This section deals with the prey capturing problem in the case of one, two and three predators. We shall show that while in the case of one or two predators the prey might escape, in the case of three predators the prey might be caught if the predators collaborate.

**The one predator case** Catching the prey in the case of one predator is equivalent with a complete overlap of the regions of influence of the prey and predator. If the overlap is not complete, the prey has an escape region represented by the shaded area in Fig. 8a. By Proposition 4.2 the centers of the regions of influence can get as close as possible, but never coincide. Hence the regions of influence never overlap after a finite number of jumps, and hence the jumping predator cannot capture the prey.

**The case of two predators** In this case the problem becomes more interesting. If \( \mathcal{R}_1, \mathcal{R}_2 \) and \( \mathcal{R} \) are the regions of influence of the predators and the prey, respectively, then the prey is caught by the predators if

\[
\mathcal{R} \subset \mathcal{R}_1 \cup \mathcal{R}_2. \quad \text{(5.13)}
\]

We shall show that the union of the regions of influence of two predators can never overlap over the entirely region of influence of the prey, unless one of the predators' position coincides with the prey's position, see Fig. 8c.

\( \Theta \) Springer
Proposition 5.1  The prey can find a strategy to escape two predators in the case of an unbounded domain.

Proof If the prey adopts one of the previous strategies, then its region of influence will never coincide with the regions of influence of any of the predators. Then the only possibility for the predators to capture the prey occurs when inclusion (5.13) holds, with $R \neq R_1$ and $R \neq R_2$. If let $C_1 = R \setminus R_1$, $C_2 = R \setminus R_2$, then

$$C_1 \cap C_2 = (R \setminus R_1) \cap (R \setminus R_2) = R \setminus (R_1 \cup R_2) = \emptyset,$$

(5.14)
i.e, the moon-shaped nonempty sets $C_1$, $C_2$ are mutually disjoint.

On the other side, since $AA' < 2\epsilon = \delta a(C_i)$, see Fig. 8a, none of the sets $C_i$ can fit into a half-disk. Then $C_i \cap \partial R$ are arcs of circle of measure larger than $\pi$, and hence they have to overlap. It follows that the sets $C_1$, $C_2$ do overlap, see Fig. 8b, which contradicts relation (5.14). Hence the inclusion (5.13) holds, which leads to the desired result. \hfill $\Box$

However, in the case of a bounded domain the prey might be caught at the boundary if the predators have a cooperative strategy.

The case of three predators  The picture looks very different in the case of three predators, though. Three circles of the same radii can always overlap over a given circle of the same radius, see Fig. 8d. There is no escape region for the prey in this case. We conclude that in the case of three predators there might be a strategy by which the prey can be caught. The diagram in Fig. 8d is known under the name of the 5-dollar coin problem, and it was discovered by the Romanian mathematician G. Tzitzeica in 1908, see Tzitzteica (1981).

6 Capturing a static camouflaged prey

This section describes a predator’s strategy to locate a well camouflaged prey, which emits signals as long as the predator is further than a certain distance from the prey.
The predator cannot see the prey, but only can hear it. It cannot directly determine either the distance to the prey or the exact direction the signals are coming from.

A real life 2-dimensional model can be considered in the case of a grasshopper singing in the night. When a predator approaches, the grasshopper stops as soon as the predator gets closer than a critical distance $R$. In this section we shall find a strategy to localize the grasshopper’s position in the case when the critical distance $R$ is unknown.

Other 3-dimensional applications can be imagined, such as a radar emitting signals as an airplane approaches. The airplane can’t determine the exact position of the radar, but can measure its own position as soon as the radar stops emitting the signals.

Let $x(t)$ be the position of the prey. The “safe” region is the disk $D(x(t), R)$. As long as the predator is outside this disk, the prey emits signals. As soon as the predator touches the boundary of the disk or enters it, the prey feels the danger and gets quiet.

When the predator gets out of the disk, the prey feels safe and starts getting noisy again. We shall find the prey’s position under the following two conditions:

- The position $x = x(t)$ is fixed (the prey does not move).
- The safety radius $R$ does not change in time.

The umbrella strategy

We shall start by making the assumption that the safety radius $R$ is known. Later we shall remove this condition. This section presents a strategy for finding possible positions of the prey.

The predator continues its trajectory $y(t)$ as long as the prey emits noise, until it reaches the stopping point $S$, and when the prey gets quiet. At this time there is a distance $R$ between the prey and the predator. Let $C_l$ and $C_r$ be two circles of radius $R$ tangent to the trajectory $y(t)$ at the point $S$. Let $G_l$ and $G_r$ be the centers of these circles. Then the prey must belong to the semicircle centered at $S$ of radius $R$ with end points $G_l$ and $G_r$, see Fig. 9a. We shall call this semicircle the umbrella of $y(t)$ centered at $S$, and we shall denote it by $U(R, S)$. The name of the strategy obviously comes from the shape of the figure.

In order to determine the position $x$ of the prey, two predator trajectories $y_1(t)$ and $y_2(t)$ will be considered with distinct stopping points $S_1$ and $S_2$. The predator needs to approach the prey from two different directions and determine the stopping points. The position $x$ of the prey is located at the intersection of the umbrellas of $y_1(t)$ and $y_2(t)$ centered at $S_1$ and $S_2$, respectively. The umbrellas intersect if the distance between the stopping points satisfies $|S_1 S_2| \leq 2R$. If the angle of the directions of the trajectories at $S_1$ and $S_2$ is relatively small, the umbrellas intersect at a unique point.

This strategy assumes the radius $R$ given. Next we shall find the position $x$ even in the case when $R$ is not provided. By the previous procedure the point $x$ belongs to the intersection of umbrellas $U(R, S_1) \cap U(R, S_2)$, for some unknown $R$. If let $R > 0$ vary as a parameter, the aforementioned intersection generates the half-line $d_{12}$ which is the perpendicular bisector of the segment $S_1 S_2$, see Fig. 9b (i.e. $d_{12}$ is perpendicular to the line segment $S_1 S_2$ and passes through its midpoint). Hence we found that $x$ belongs to the line $d_{12}$.
Next we consider another predator trajectory $y_3(t)$ with the stopping time $S_3$. According to the previous method, the point $x$ belongs to the perpendicular bisector $d_{23}$ of the segment $S_2S_3$, and hence $x = d_{12} \cap d_{23}$. We have obtained that the prey must be positioned at the intersection point of perpendicular bisectors of the triangle $S_1S_2S_2$. It is well-known from geometry that this point always exist and it is unique as long as the points $S_1$, $S_2$, $S_2$ are distinct and non-collinear.

### 7 Discussion

In Sect. 1 the prey and the predator are confined to move inside of a 2-dimensional disk, and their positions are represented by points. The results still hold true if the disk is replaced by a convex bounded domain. The result fails for concave domains, since in this case the predator might not see the prey all the time due to possible corners. Another important case occurs for unbounded domains with border, such as a half-plane. This models, for instance, a dog pursuing a horse on a sea-shore. The horse is constrained to move only on one side of the boundary. The result of Proposition 1.1 still holds in this case, i.e., the dog can get as close as possible to the horse.

The problem becomes more interesting if the prey and the predator are moving across a surface, such as a spherical dome, a cylinder, or a cube. We can think of a spider chasing an ant, both being constrained to move on the interior walls of a room. Since the speed is constant, the shortest time trajectory is also distance minimizing. If they are on the same wall, then the spider will choose to attack by taking the shortest trajectory to the ant, which in this case is a straight line segment. If the wall is not flat (i.e. it is a curved surface), then the straight trajectory is replaced by the curve of the shortest distance, called geodesic. For instance, in the case of a spherical wall, the geodesic is an arc of a great circle. Going back to the case of a cubical room, assume the agents have the initial positions on distinct walls, see Fig. 10a. Getting from one wall to another is done under a certain angle. The situation is represented in more detail in Fig. 10b.
The predator and the prey are located at the points $B$ and $D$, respectively, on two distinct planes $\alpha$ and $\beta$, which intersect over the line $AC$. Let $O, Q \in AC$ such that $BO$ and $DQ$ are perpendicular on $AC$. Using triangle inequality yields

$$|BP| + |PD| \leq |BO| + |OP| + |PQ| + |QD| = |BO| + |OQ| + |QD|,$$

it follows that the length of the broken path $f(P) = |BP| + |PD|$ attains its maxima for $P \in \{O, Q\}$. Since $f$ is continuous, it must have a minimum at a certain point $P$ in between $O$ and $Q$. The point $P$ is obtained by developing the dihedral angle $\{\alpha, \beta\}$ onto a plane and joining the images of $B$ and $D$ by a straight line. The point where this line intersects the common edge is the image of $P$. Hence the point $P$ has the property that $\angle OPB = \angle QPD$. This implies the following similar right triangles $\Delta OPB \sim \Delta QPD$. If denote $a = |BO|$, $b = |DQ|$, $u = |OP|$, $v = |PQ|$, and $d = |OQ|$, the proportionality ratios yield

$$\frac{a}{b} = \frac{u}{v} \implies u = \frac{ad}{a + b}. \quad (7.15)$$

Using (7.15) one can find the broken line between the predator and prey on a cubical room, provided the agents belong to adjacent faces. The case when the agents belong to opposite walls is more complicated; the uniqueness of the optimal path in this case might fail, sometimes having even infinitely many optimal paths (such as in the case of centers of two opposite walls of a cubical room).

Even if these problems are looking similar with the problem for a plane bounded domain, the result of Proposition 1.1 does not hold true here. If the prey and the predator move at the same pace on the interior side of a cubical room, or a sphere on a cyclic trajectory, then the prey will never be caught. Furthermore, the distance between them can be kept constant. This is a similar situation with the one when the agents move on an infinite plane. The result is not surprising since both the sphere and the cube can be mapped onto an entire plane.
The problem is also natural in the 3-dimensional setting too. Birds flying above the
ground or fish swimming under the water surface are only two examples. The result
of Proposition 1.1 still holds true in this case, but we omit its proof.

Another interesting analysis is obtained if we drop the agents’ synchronicity, by
introducing a time lag \( \tau \) needed for the predator to adjust its position. We shall assume
\( \tau > 0 \) constant, but other more elaborated models can be developed by assuming
that \( \tau \) depends on the distance or velocity. This can be interpreted by saying that the
 predator can access at time \( t \) only the information coming from the prey available until
time \( t - \tau \). With the aforementioned notations, the prey’s position \( y(t) \) depends on
\( x(t - \tau) \) and the Eq. (1.1) becomes

\[
y(t) + cy'(t) (\rho(t) - v\tau) = x(t - \tau).
\]

If the agents’ speeds are unitary, \( v = 1 \), then we may scale \( c = 1 \). The term \( \rho(t) - v\tau \)
represents the distance between \( y(t) \) and \( x(t - \tau) \). According to Proposition 1.1, the
 predator \( y(t) \) can get as close as possible to the prey’s position \( x(t - \tau) \). Since the
real prey position \( x(t) \) is always situated at a distance \( \tau \) from \( x(t - \tau) \) (assuming
the velocity \( v = 1 \)), then the predator will never catch the prey. However, if the
 predator’s speed exceeds the prey’s speed, then the prey catches the prey.

The analysis can be refined if assume the agents have distinct masses. For instance, if
the mass of the predator is much larger than the one of the prey, even if they have the
same velocities, the predator cannot make sharp turns to follow the prey and hence
this will be difficult to catch. This occurs for instance in the case of a large condor
trying to catch a small robin or an ox pursuing a dog. If the mass of the prey is larger
than the one of the predator, the best the prey can do is to run straight, refraining from
making turns.

Our assumption that the the predator’s strategy is to have the direction pointing
always to the prey’s position is the most natural and instinctive. However, more effi-
cient algorithms can be imagined with applications to rocket interception in space, for
instance. If the prey is a spaceship and the predator is a chasing rocket, then the rocket
should adjust its direction by anticipating the future spaceship’s position, assuming its
velocity remains the same. If the predator observes the prey at the instance \( t \) having
the position \( x(t) \) and velocity \( x'(t) \), then it anticipates the position of the prey at the
future time \( t + T \) to be \( x(t) + x'(t)T \). Pointing toward this anticipated position, the
 predator’s position \( y(t) \) should satisfy the equation

\[
y(t) + y'(t)T = x(t) + x'(t)T.
\]

This leads to \( T = \rho(t)/|x'(t) - y'(t)| \). In this case the prey’s strategy should be to
make unpredictable turns.

The second section of the paper deals with different strategies involving one prey
pursued by two predators. Here the type of the domain is critical in applying the proper
strategy. If the domain is an infinite plane, then the prey should follow a straight tra-
jectory which is the line perpendicular to the midpoint of the segment between the
predators. This works under the hypothesis that both predators have the same speed. If
The prey is caught only if $w > v$. The prey’s position $x(t)$ tends to the center of the circle if this condition is dropped, the prey’s optimal trajectory is no longer the aforementioned perpendicular line.

If the motion is considered on a boundless plane, the prey won’t be caught by its predators, unless they have a larger speed. This is represented in Fig. 11a. If the prey and predator speeds are denoted by $v$ and $w$, the distance between predators by $2d$, and the catching time by $T$, then $w^2T^2 = v^2T^2 + d^2$, from where $T = d / \sqrt{w - v}$. It is worth noting that if $w = v$, then the catching time $T$ becomes infinite. More challenging problems can be obtained if allow for the velocity of the predators to be variable, for instance to be decreasing in time.

Since we discussed only the case of non-collaborative strategy of predators, we shall consider next the left over case versus the dimension of the space. In one dimension, it is obvious that the prey is always caught if the predators come from opposite directions and the prey is in between them, and this occurs regardless of speeds. In two dimensions, the predators can always form a triangle $y_1(t)y_2(t)y_3(t)$. They can cooperate such that the prey $x(t)$ is situated inside the triangle. Then the prey will apply a similar strategy like in the case of two predators, which leads to equal distances to its predators. A such point always exists for a triangle and it is known to be the center of its circumcircle, see Fig. 11b. The narrower the predators’s circumcircle, the less room for the prey. In the long run the prey is caught.

In the case of three dimensions, our hunch says that 4 flying predators can catch a flying prey. However, it is known from geometry that any 4 points do not always belong to a sphere, so there might be no point with equal distance from each predator.

One possible future direction of study is to investigate the strategy of $n$ preys and $m$ predators. One possible question here is what is the relationship between the numbers $n$ and $m$ in order for the predators to capture the all preys?

Section 3 describes a min–max strategy in the case of unbounded domains. It is worth noting that the strategy fails in the neighborhood of boundary points or corner points. For instance a bird can catch a jumping frog easier if the frog is first pushed into a corner. We also note that if the jumps are considered infinitesimally small, the motion of the agents becomes continuous.
Geometric optimization for prey–predator strategies

![Fig. 12](image)

The regions of influence considered so far have a circular shape, but this is not a rule. Certain landscape obstructions can change the shape of these regions drastically, see Fig. 12.

The formulas developed in Sect. 4 have a computational relevance. They can easily be implemented on a computer to simulate the movement of the agents. Making the radius small, the simulation approximates a smooth move into a plane. This approach is useful while exact formulas are not on hand.

The main idea of Proposition 5.1 is that two equal disks cannot cover completely a given disk of the same size, unless one of them overlaps the given disk. In the 3-dimensional case the regions of influence are spheres. A similar property works for spheres: two equal spheres cannot overlap onto a given sphere of the same radius, and hence Proposition 5.1 works also in the 3-dimensional space. It worth noting that a 3-dimensional diagram similar to Fig. 10d does not work if the circles are replaced by spheres.

Section 6 discusses a strategy of catching a camouflaged prey. One critical assumption here is that the prey does not move. It would be more natural to assume that the prey changes its position between signals emission. Consider a certain species of prey, such as a fish or an insect, which emits a fluorescent light as long as there are no predators around it, and which becomes invisible at once when a predator reaches a certain critical distance $R$ from it. This distance can be regarded as the maximum horizon at which the fish can detect its predators. Let $\rho$ be the maximum distance from which the predator can see its prey. It makes sense to assume $\rho > R$, since otherwise the predator will never get the chance to even detect the prey. From the moment the prey becomes invisible, it will also try to move as far away as possible from the spot where it was detected at its maximum speed. If it applies the strategy described in Sect. 1, the prey will move collinearly with the predators’ position and its former spot. Assuming the moving distance $d > 0$ is known. Since the prey’s position before the move belongs to an umbrella, see Fig. 9a, after moving a distance $d$ in the opposite direction of the predator it will belong to the convex hull of circles centered on the umbrella, see Fig. 13. Intersecting two of these convex hulls provides the position of the prey. This techniques assumes $R$ and $d$ given. It would be interesting to find an algorithm in the case when these data are unknown.
The prey belongs to the convex hull of the circles of radius $d$ centered on the umbrella of the stopping point $S$.

A similar problem can be imagined where the predator can sniff its prey from a distance $\rho$, while the prey can sniff predators from a distance $R$, with $\rho > R$. The predator’s sniffing region is the disk $D(y(t), R)$. The prey is detected when $x(t) \in D(y(t), R)$.

Since diffusion is not directional, in the sense that the prey cannot know the direction the smell comes from, the best the prey can do is to move around and overlap several sniffing regions until it gets as close as the prey enters its visual field.

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References


