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Non-Connectivity example in subRiemannian geometry†

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The article deals with a step 2 example of subRiemannian geometry which locally resembles the Grusin example. We show that while locally the connectivity by geodesics holds and it behaves very similar to the Grusin case, globally the connectivity does not hold.

\textbf{Keywords:} subRiemannian geodesic; Hamiltonian equations; Elliptic functions

\textbf{AMS Subject Classifications:} 53C17; 53C22; 35H20

1. Introduction

Suppose we are given \(m\) linearly independent vector fields \(X_1, \ldots, X_m\) on an \(n\)-dimensional manifold \(M_\omega\), \(m \leq n\). We shall refer to \(X_1, \ldots, X_m\) and their linear combinations as ‘horizontal’ vector fields and a curve with horizontal tangents will be called a horizontal curve. We assume that the set of horizontal vector fields \(X = \{X_1, \ldots, X_m\}\) is an orthonormal set with respect to a metric defined on \(\text{span } X\). If \(m = n\), this yields a Riemannian metric on \(M_\omega\). Let \(X^*_j\) denote the vector field adjoint to \(X_j\) with respect to the obvious volume element, then

\[\Delta = -\frac{1}{2} \sum_{j=1}^{n} X^*_j X_j\]

is the usual Laplace–Beltrami operator. The Newtonian potential is

\[N(x, y) = \frac{1}{(2-n)|S_\omega(y)|d(x, y)^{n-2}}, \quad n > 2,\]

where \(|S_\omega(y)|\) is the surface area of the induced unit ball with centre \(y\), and \(d(x, y)\) is the Riemannian distance between \(x\) and \(y\). Then

\[\Delta_N(x, y) = \delta(x - y) + O(d(x, y)^{-n+1}),\]

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†Dedicated to Chung Chun Yang on the occasion of his 65th birthday.
in other words, the fundamental solution differs from \( N(x, y) \) be a negligible error, the singularity is weaker than the singularity of the Dirac \( \delta \)-function. We recall that the fundamental solution \( K(x, y) \), which a distribution satisfying

\[
\Delta_x K(x, y) = \delta(x - y),
\]

is the kernel of the integral operator inverse to the operator \( \Delta \),

\[
\Delta_x \int K(x, y)f(y)dy = f(x).
\]

In this case, we call the manifold \( \mathcal{M}_n \) of step 1.

When \( m < n \), we shall assume that \( X_1, \ldots, X_m \) together with a finite number of Lie brackets of \( X_1, \ldots, X_m \) generate the tangent bundle \( TM_n \). Then Chow’s theorem \([1]\) says that between any two points, there is a \( C^1 \)-piecewise horizontal curve. To see how remarkable Chow’s theorem is, note that given two vector fields

\[
X_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x_2}
\]

in \( \mathbb{R}^3 = \{x = (x_1, x_2, x_3)\} \), there is no horizontal curve joining any two points which have different \( x_3 \)-components. Therefore, we always assume that the vector fields \( X_1, \ldots, X_m \) satisfy the bracket generating condition. If one bracket suffices, we call \( \mathcal{M}_n \) of step 2. Otherwise, we say it is of higher step. Let \( \gamma \) be a horizontal curve between two given points. Its velocity is

\[
\dot{\gamma} = \sum_{j=1}^{m} a_j(x)X_j.
\]

As usual,

\[
\ell(\gamma) = \int \sqrt{\sum_{j=1}^{m} a_j^2(x)}dx
\]

is the length of \( \gamma \). By minimizing the lengths of horizontal curves between \( x, y \in \mathcal{M}_m \), we obtain the distance of \( x \) from \( y \) and hence a geometry on \( \mathcal{M}_n \) which we shall call \textit{subRiemannian}. Set

\[
X_j = \sum_{k=1}^{n} a_{jk}(x) \frac{\partial}{\partial x_k}, \quad j = 1, \ldots, m.
\]

Then

\[
H = \frac{1}{2} \sum_{j=1}^{m} \left( \sum_{k=1}^{n} a_{jk}(x)\xi_k \right)^2
\]

is the Hamiltonian function associated with the subelliptic operator \( \frac{1}{2} \sum_{j=1}^{m} X_j^2 \) and it is defined on the cotangent bundle \( T^*\mathcal{M}_n \). A bicharacteristic curve \( (x(s), \xi(s)) \in T^*\mathcal{M}_n \) is a solution of the Hamiltonian system of differential equations:

\[
\begin{align*}
x_j(s) &= H_{\xi_j}, \\
\dot{\xi}_j(s) &= -H_{x_j},
\end{align*}
\]

with the boundary conditions

\[
x_j(0) = x_j^{(0)}, \quad x_j(\tau) = x_j, \quad j = 1, \ldots, n,
\]
for given end-points $x^{(0)}$, $x \in \mathcal{M}_n$; one may think of $\tau$ as time. The projection $x(s)$ of the bicharacteristic curve on $\mathcal{M}_n$ is a geodesic. SubRiemannian geometry is quite different from Riemannian geometry. Below we review a few of these differences:

1. Every point $O$ of a Riemannian manifold is connected to every other point in a sufficiently small neighbourhood by only one geodesic. On a subRiemannian manifold there will be points arbitrarily near $O$ which are connected to $O$ by an infinite number of geodesics (see e.g. [2–4]). This strange phenomenon was first pointed out by Strichartz [5], and it brings up the question of what ‘local’ means in subRiemannian geometry. Control theorists (see e.g. [6,7]) studying subRiemannian examples noticed that the Riemannian concepts of cut locus and conjugate locus behave badly in a subRiemannian context.

2. In Riemannian geometry the unit ball is smooth. In subRiemannian geometry, among the many distances, there is a shortest one, often referred to as the Carnot–Carathéodory distance. In subRiemannian geometry the Carnot–Carathéodory unit ball is singular.

3. The exponential map is smooth in Riemannian geometry, but often singular in subRiemannian geometry. The singularities occur at points connected to an ‘origin’ by an infinite number of geodesics. These singular points constitute a submanifold whose tangents yield the ‘missing directions’, which are the directions in $T\mathcal{M}_n$ not generated by the horizontal directions.

For instance, one well-known example is

$$X_1 = \frac{\partial}{\partial x_1} + 2k x_2 (x_1^2 + x_2^2)^{k-1} \frac{\partial}{\partial x_0}, \quad X_2 = \frac{\partial}{\partial x_2} - 2k x_1 (x_1^2 + x_2^2)^{k-1} \frac{\partial}{\partial x_0} \quad (1)$$

in $\mathbb{R}^3 = \{ (x_0, x_1, x_2) \}$. It is easy to see that the vector fields $X_1$ and $X_2$ satisfy the bracket generating condition, i.e. $X_1$, $X_2$ and their iterated Lie brackets span the tangent bundle $T(\mathbb{R}^3)$ at each point of $\mathbb{R}^3$. For $k = 1$, one obtains the Heisenberg vector fields, which is step 2 everywhere. Furthermore, $X_1$ and $X_2$ are left translation invariant under the following non-commutative group law on $\mathbb{R}^3$:

$$(x_0, x_1, x_2) \circ (y_0, y_1, y_2) = (x_1 + y_1, x_2 + y_2, x_0 + y_0 + (y_1 x_2 - x_1 y_2)). \quad (2)$$

In this case we can prove that for any two points in $\mathbb{R}^3$ can be connected by a smooth horizontal curve [8,9]. However, this property does not hold in general. The relationship with hypoellipticity of the sum of squares operators which satisfy the bracket generating condition is done in [10].

In this article, we consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = \sin x \frac{\partial}{\partial y}$ on $\mathbb{R}^2 = \{ (x, y) \}$. Since we have

$$Z = [X, Y] = \cos x \frac{\partial}{\partial y}, \quad [X, Z] = -Y, \quad [Y, Z] = 0,$$

the vector fields $\{X, Y\}$ span the tangent space of $\mathbb{R}^2$ at each point of $\mathbb{R}^2\setminus S$, where

$$S = \bigcup_{n \in \mathbb{N}} \{ (n \pi, x_2); x_2 \in \mathbb{R} \},$$

and $\{X, Y, Z\}$ span the tangent space at the points of $S$. In this case the model is step 2 along $S$, and step 1 otherwise. Here we shall investigate the geometry induced by $\{X, Y\}$ from the connectivity by geodesics point of view.
Similar vector fields have been considered in earlier work where different terms \( x^k \) were substituted for the coefficient \( \sin x \) in the vector field \( Y \). The case when \( k = 1 \) leads to the Grushin model and was treated in [11] using trigonometric functions. The case \( k = 2 \) was solved by means of elliptic functions in [3]. All the cases \( k \geq 3 \) are more difficult to solve and need the use of hypergeometric functions. We shall leave that for a forthcoming paper. The present example is important because \( \sin x = x - \frac{x^3}{3!} + \cdots \) contains information about all the odd powers of \( x \), and because in this case the geodesics can be expressed in closed form. We also want to point out that the same method presented in this article can be applied to vector fields \( \tilde{X} = \frac{\partial}{\partial x} \) and \( \tilde{Y} = \cos x \frac{\partial}{\partial y} \) on \( \mathbb{R}^2 \) to obtain similar results.

The case of similar vector fields in three dimensions were also studied in [2,4,8,9]. In the following we shall recall a few notions of elliptic functions which will be used in the sequel. For more advance treatment, the interested reader can consult the book of Lawden [12]. The integral

\[
 z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}, \quad |k| < 1
\]

is called an elliptic integral of the first kind. The integral exists if \( w \) is real and \( |w| < 1 \).

Using the substitution \( t = \sin \theta \) and \( w = \sin \phi \)

\[
z = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}.
\]

If \( k = 0 \), then \( z = \sin^{-1} w \) or \( w = \sin z \). By analogy, the above integral is denoted by \( \sin^{-1}(w; k) \), where \( k \neq 0 \). \( k \) is called the modulus. Thus

\[
z = \sin^{-1} w = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}.
\]

The function \( w = \sin z \) is called a Jacobian elliptic function.

By analogy with the trigonometric functions, it is convenient to define other elliptic functions

\[
\begin{align*}
\text{cn} z &= \sqrt{1 - \sin^2 z}, \quad \text{dn} z = \sqrt{1 - k^2 \sin^2 z}.
\end{align*}
\]

A few properties of these functions are

\[
\begin{align*}
\text{sn}(0) &= 0, \quad \text{cn}(0) = 1, \quad \text{dn}(0) = 1, \\
\text{sn}(-z) &= \text{sn}(z), \quad \text{cn}(-z) = \text{cn}(z), \\
\frac{d}{dz} \text{sn} z &= \text{cn} z \text{dn} z, \quad \frac{d}{dz} \text{cn} z = -\text{sn} z \text{dn} z, \quad \frac{d}{dz} \text{dn} z = -k^2 \text{sn} z \text{cn} z,
\end{align*}
\]

\[-1 \leq \text{cn} z \leq 1, \quad -1 \leq \text{sn} z \leq 1, \quad 0 \leq \text{dn} z \leq 1.\]

Let

\[
K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}
\]

be the complete Jacobi integral. Then, as real functions, the elliptic functions \( \text{sn} \) and \( \text{cn} \) are periodic functions of principal period \( 4K \), see Figure 1.
2. General properties of geodesics

The geodesics can be defined and described using the Hamiltonian formalism as in the following. We shall assume the space $\mathbb{R}^2$ endowed with a metric in which the vector fields $X$ and $Y$ are orthonormal. The Hamiltonian associated with the above vector fields is the principal symbol of the subelliptic operator $L = \frac{1}{2}(X^2 + Y^2)$

$$H(x, y; \xi, \theta) = \frac{1}{2}\xi^2 + \frac{1}{2}(\sin x)^2\theta^2.$$ 

The Hamiltonian system of bicharacteristics is

$$\begin{align*}
\dot{x} &= H_\xi = \xi \\
\dot{y} &= H_\theta = (\sin x)^2\theta \\
\dot{\xi} &= -H_x = -\theta^2 \sin x \cos x \\
\dot{\theta} &= -H_y = 0 \Rightarrow \theta = \text{constant}.
\end{align*}$$

The geodesic parametrized by $[0, \tau]$ which connects the points $(x_0, y_0)$ and $(x, y)$ is the projection of the bicharacteristics on the $(x, y)$-plane satisfying the boundary conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad x(\tau) = x, \quad y(\tau) = y.$$ 

Case $\theta = 0$: In this case $\ddot{x} = 0$ and $\ddot{y} = 0$ with the solution

$$x(s) = x_0 + \frac{s}{\tau}(x - x_0), \quad y(s) = y_0 = y.$$ 

**Proposition 2.1** Let $y_0 = y$. There is a unique geodesic joining the points $(x_0, y_0)$ and $(x, y)$. This is a straight line given by formulae (3).

**Proof** Integrating in the second equations of the Hamiltonian system yields

$$y = y_0 + \theta \int_0^\tau \sin^2 x(s) \, ds.$$ 

Since $y = y_0$ and the integral does not vanish, it follows that $\theta = 0$ and hence formulae (3) hold.
Case $\theta \neq 0$: The first and the third equations of the Hamiltonian system yield the following ordinary differential equation for $x$

$$\ddot{x} = \dot{x} = -\theta^2 \sin x \cos x = -\frac{\theta^2}{2} \sin(2x).$$

Substituting $u = 2x$ we get

$$\ddot{u} = -\theta^2 \sin u,$$  \hspace{1cm} (4)

which is known as the *simple pendulum equation*. The variable $u$ denotes the angle made by the pendulum string with the downward vertical. The constant $\theta$ is given by $\theta^2 = g/\ell$, which is the quotient between the gravitational acceleration and the length of the string. The solution $u(s)$ of Equation (4) can be represented as

$$\sin \frac{u}{2} = \kappa \text{sn}(\theta s, k),$$

where $k = \sin \frac{\alpha}{2}$ and $\alpha$ denotes the maximum amplitude of the pendulum, [12, p. 115]. The parameter $s$ denotes the time taken for the pendulum’s bob to move from its lowest position to a position at which the string makes the angle $u(s)$ with the vertical. Then

$$\sin x(s) = \kappa \text{sn}(\theta s, k),$$  \hspace{1cm} (5)

where we considered the initial condition $x(0) = 0$. This means $\sin x(s)$ oscillates with period $4K/\theta$, where

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - \kappa^2 t^2)}}.$$  \hspace{1cm} (6)

Standard trigonometric formulae yield

$$\cos x(s) = \begin{cases} 
\sqrt{1 - k^2 \sin^2(\theta s, k)} = \text{dn}(\theta s, k), & x(s) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}] \cup \ldots \\
-\sqrt{1 - k^2 \sin^2(\theta s, k)} = -\text{dn}(\theta s, k), & \text{otherwise}.
\end{cases}$$

Hence we obtain

$$\tan x(s) = \frac{\sin x(s)}{\cos x(s)} = \begin{cases} 
k \frac{\sin(\theta s, k)}{\text{dn}(\theta s, k)} = k \text{sd}(\theta s, k), & x(s) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}] \cup \ldots \\
-k \frac{\sin(\theta s, k)}{\text{dn}(\theta s, k)} = -k \text{sd}(\theta s, k), & \text{otherwise}.
\end{cases}$$

Then the $x$-component of the geodesic which starts at $x(0) = 0$ is given by

$$x(s) = \arctan \left( k \text{sd}(\theta s, k) \right) \text{ for } |x(s)| < \frac{\pi}{2}.$$  \hspace{1cm} (7)

In the following we state a non-connectivity result.
**Proposition 2.2** Let \( y_0, y \) arbitrary fixed. If \( |x| > \frac{\pi}{4} \), then there is no geodesic joining the points \((0, y_0)\) and \((x, y)\).

**Proof** Since \( \text{sd}(,) \) is periodic and bounded above and below by \(+1\) and \(-1\), it follows from Equation (7) that \( x(s) \) is bounded with \( |x(s)| \leq \arctan(k) \). Since \( k \leq 1 \), then \( |x(s)| \leq \pi/4 \), i.e. the geodesic starting at the origin is contained in the strip between the vertical lines \( x = \pm \frac{\pi}{4} \). Hence the geodesics starting at \((0, y_0)\) cannot reach the points situated outside the strip \( |x| \leq \frac{\pi}{4} \).

In the following we shall find the \( y \)-component of the geodesic. Integrating in the second equation of the Hamiltonian system

\[
\dot{y} = \theta (\sin x)^2 = \theta k^2 \text{sn}^2(\theta s, k)
\]

we obtain

\[
y(s) = y(0) + \theta k^2 \int_0^s \text{sn}^2(\theta s, k) \, ds
\]

\[
= y(0) + \theta \left( s - \int_0^s \text{dn}^2(\theta s, k) \, ds \right)
\]

\[
= y(0) + \theta s - \int_0^{\theta s} \text{dn}^2 v \, dv
\]

\[
= y(0) + \theta s - E(\theta s, k), \quad (8)
\]

where \( E(,) \) is the Jacobi epsilon function given by

\[
E(u, k) = \int_0^u \text{dn}^2 v \, dv, \quad (9)
\]

[12, p. 62]. Denote by

\[
E = E(K) = \int_0^K \text{dn}^2 v \, dv,
\]

where \( K \) is given in (6). We note that \( K - E > 0 \).

**Lemma 2.3** The function \( y(s) \) is increasing for \( \theta > 0 \) and decreasing for \( \theta < 0 \). Furthermore, each increment of the parameter \( s \) by \( 2K/\theta \) produces an increase equal to \( 2(K - E) \) in the function \( y(s) \)

\[
y\left( s + \frac{2nK}{\theta} \right) = y(s) + 2n(K - E), \quad \forall n \in \mathbb{Z}.
\]

In particular, if \( y(0) = 0 \), then

\[
y\left( \frac{2nK}{\theta} \right) = 2n(K - E), \quad \forall n \in \mathbb{Z}.
\]

**Proof** Differentiating in (8) yields

\[
y'(s) = \theta - E'(\theta s) \theta = \theta (1 - \text{dn}^2 u).
\]

Since \( \text{dn}^2 u \leq 1 \), the first part follows easily.
Using the formula \( E(u + 2nK, k) = E(u, k) + 2nE, \) [12, p. 64], substituting in (8) yields
\[
y(s + \frac{2nK}{\theta}) = y(0) + \theta s + 2nK - E(\theta s + 2nK, k)
\]
\[
= y(0) + \theta s + 2nK - E(\theta s) - 2nE
\]
\[
= y(s) + 2n(K - E).
\]
The last part is an obvious consequence of the above relation.

**2.1. Geodesics connecting the origin and the point \((0, y)\)**

In this section, we shall investigate the geodesics joining the origin \((0, 0)\) with a point \((0, y)\). We shall assume in the following calculations \(\theta > 0\). Since \(y(0) = 0\), Lemma 2.3 yields \(y > 0\) (and also if \(\theta < 0\), then \(y < 0\)).

We shall assume the geodesic parametrized by the interval \([0, \tau]\). Since \(x(0) = x(\tau) = 0\), it follows from (5) that \(sn(\theta \tau, k) = 0\) and hence
\[
\theta \tau = 2mK, \quad m = 1, 2, \ldots
\]
where \(K\) is the complete elliptic integral introduced by (6). For any natural number \(m\) we have a corresponding \(\theta_m = \frac{2mK}{\tau}\) for which the \(x\)-component becomes
\[
x_m(s) = \arctan(k sd(\theta_ms, k)) = \arctan \left( k sd \left( \frac{2mK}{\tau}, s, k \right) \right), \quad m = 1, 2, \ldots \tag{10}
\]
Substituting \(\theta_m\) in (8) yields the corresponding expression of \(y(s)\)
\[
y_m(s) = \frac{2mK}{\tau} s - E \left( \frac{2mK}{\tau} s, k \right), \quad m = 1, 2, \ldots \tag{11}
\]
In the following, we shall find an equation for \(k\) in terms of the boundary conditions. Using the last part of Lemma 2.3 yields
\[
y = y(\tau) = y \left( \frac{2mK}{\theta_m} \right) = 2m(K - E),
\]
and hence
\[
K - E = \frac{y}{2m}, \quad m = 1, 2, \ldots
\]
where the left side term is a function of \(k\)
\[
\psi(k) = K(k) - E(k)
\]
for \(k \in (0, 1)\). The function \(\psi(k)\) has the following properties:

(i) \(\psi(k) > 0\), with \(\lim_{k \to 0} \psi(k) = 0\).

(ii) \(\psi(k)\) is increasing.

(iii) \(\psi(k)\) has a vertical asymptote at \(k = 1\). See also Figure 2.
Since $y > 0$, for every $m \in \{1, 2, \ldots \}$ there is a unique $k_m$ such that

$$\psi(k_m) = \frac{y}{2m}. \quad (12)$$

**Proposition 2.4**

(i) There are infinitely many geodesics joining the origin $(0, 0)$ with the point $(0, y)$, $y \neq 0$.

(ii) The geodesics are parametrized by the solutions of Equation (12).

(iii) For each $m = 1, 2, \ldots$ the corresponding geodesic is given by

$$x_m(s) = \arctan \left( k_m \frac{2mK(k_m)}{\tau} s, k_m \right),$$

$$y_m(s) = \frac{2mK(k_m)}{\tau} s - E \left( \frac{2mK(k_m)}{\tau} s, k_m \right).$$

(iv) We have

$$\lim_{m \to \infty} \max_{s \in [0, \tau]} |x_m(s)| = 0,$$

i.e. the $m$-th geodesic is contained in a vertical strip whose width tends to zero as $m$ increases unbounded.

**Proof** The parts (i) and (ii) follow from the previous discussion. Substituting $k = k_m$ in Equations (10) and (11) yield formulae in (iii). Since $k_m \to 0$ as $m \to \infty$, see Equation (12), the estimation

$$|x_m(s)| = |\arctan \left( k_m \frac{2mK(k_m)}{\tau} s, k_m \right)| \leq |\arctan(k_m)|$$

leads to (iv).
2.2. The length of geodesics

In the following, we shall find the lengths of geodesics joining the origin \((0, 0)\) and the point \((0, y)\), \(y > 0\). The length will be computed with respect to the metric on \(\mathbb{R}^2\) in which the vector fields \(X = \partial_x\) and \(Y = \sin x \partial_y\) are orthonormal.

Let \(\ell\) denote the length and \(H\) be the Hamiltonian along the geodesic. Since \(\ell = \sqrt{2H}\) the first concern is to compute the value of the Hamiltonian \(H\) in terms of the boundary value \(y\).

Using that \(\text{cn}'(s, k) = -\text{sn}(s, k) \text{dn}(s, k)\), from one of the Hamiltonian equations we have

\[
\dot{\xi} = -\theta^2 \sin x \cos x = -\theta^2 \text{ksn}(\theta s, k) \text{dn}(\theta s, k)
\]

\[
= k\theta \frac{d}{ds} \text{cn}(\theta s, k) \implies \xi(s) = k\theta \text{cn}(\theta s, k) + C,
\]

with the integration constant \(C = \xi(0) - k\theta = \dot{x}(0) - k\theta\). Differentiating in the expression \(\sin x(s) = k\text{sn}(\theta s, k)\) yields \(\cos x(s) \dot{x}(s) = k\theta \text{cn}(\theta s, k) \text{dn}(\theta s, k)\). Taking \(s = 0\) and using \(x(0) = 0\) yields \(\dot{x}(0) = k\theta \text{cn}(0) \text{dn}(0) = k\theta\). Hence the integration constant is \(C = 0\) and it follows that

\[
\xi(s) = k\theta \text{cn}(\theta s, k).
\]

On the other side, relation (5) yields

\[
(\sin x(s))^2 \theta^2 = k^2 \text{sn}^2(\theta s, k) \theta^2.
\]

Using relations (13) and (14) together with the identity \(\text{sn}^2 u + \text{cn}^2 u = 1\), the value of the Hamiltonian along the geodesics becomes

\[
H = \frac{1}{2} \xi^2(s) + \frac{1}{2} (\sin x(s))^2 \theta^2 = \frac{1}{2} k^2 \theta^2.
\]

Hence the length of the geodesic is \(\ell = \sqrt{2H} = k|\theta|\). Substituting \(\theta = \theta_m\) and \(k = k_m\), we arrive at the following result.

**Proposition 2.5** The lengths of the geodesics joining the origin with the point \((0, y)\), \(y > 0\) are

\[
\ell_m = \frac{2mK(k_m)}{\tau} k_m,
\]

where \(k_m\) is the solution of Equation (12).

**Proposition 2.6** The lengths of the geodesics are unbounded from above, i.e.

\[
limit_{m \to \infty} \ell_m = \infty.
\]

**Proof** The proof has geometric flavor, see Figure 3. We note that \(\lim_{m \to \infty} k_m = 0\) and \(\lim_{m \to \infty} K(k_m) = K(0) = \frac{\pi}{2}\). Hence, in order to show (15), it suffices to prove

\[
limit_{m \to \infty} m \cdot k_m = \infty.
\]
With the notation as in Figure 3 we have
\[ m \cdot k_m = \frac{k_m}{\tan \phi_m} = \frac{1}{\tan \phi_m}, \]
If \( m \to \infty \), then \( k_m \to 0 \) and hence \( P_m \to O \) and \( \tan \phi_m \to 0 \), since the secant \( OP_m \) approaches the tangent at the origin, which is the \( x \)-axis. Then \( \tan \phi_m \to 0 \), which proves relation (16).

2.3. Geodesics between the origin and \((x, y), x \neq 0\)
We have seen by Proposition 2.2 that there are no geodesics joining the origin with points \((x, y), |x| > \pi/4\). In this section, we shall deal with the existence of the geodesics between the origin and points \((x, y), |x| \leq \pi/4\). The following result will be useful in the proof of the main result of the section:

**Lemma 2.7** Consider the function \( W(u, k) = u - E(u, k) \), where \( E \) is the Jacobi function defined by (9). The following relations hold:

(i) \( W(\cdot, k) \) is increasing in the variable \( u \) with inflexions at \( u = 2mK(k), m \in \mathbb{Z} \). 
(ii) \( W(-u, k) = -W(u, k) \), i.e. \( W \) is odd in the variable \( u \). 
(iii) \( W \) is unbounded with \( \lim_{u \to \pm \infty} W(u) = \pm \infty \).

**Proof**
(i) Differentiating yields
\[
\frac{d}{du} W(u, k) = 1 - \text{dn}^2(u, k) = 1 - (1 - k^2 \text{sn}^2(u, k)) = k^2 \text{sn}^2(u, k) \geq 0.
\]
The identity is reached for \( u = 0, \pm 2K, \pm 4K, \ldots \)
(ii) It follows from the fact that \( E(-u, k) = \int_0^{-u} \text{dn}^2(v, k) \, dv = -\int_0^u \text{dn}^2(v, k) \, dv = -E(u, k) \).
(iii) Taking the limit \( u \to \pm \infty \) in the relation

\[
W(u, k) = u - E(u, k) = \int_0^u (1 - \text{dn}^2 v) dv = k^2 \int_0^u \text{sn}^2 v dv
\]

leads to the desired result. \( \blacksquare \)

The above lemma states that the function \( W(\cdot, k) : \mathbb{R} \to \mathbb{R} \) is invertible for any \( k \in (0, 1) \).

In order to find the number of geodesics joining the points \((0, 0)\) and \((x, y)\), we shall find the number of pairs \((\theta, k)\) which satisfy the conditions

\[
x(\tau) = x, \quad y(\tau) = y.
\]

Substituting \( s = \tau \) in (5) and (8) yields

\[
\sin x = k \text{sn}(\theta \tau, k) \quad y = \theta \tau - E(\theta \tau, k) = W(\theta \tau, k) \implies \theta \tau = W^{-1}(y, k).
\]

Substituting the second relation in the first one yields an equation in the variable \( k \) only

\[
\sin x = k \text{sn}(W^{-1}(y, k), k).
\]

The number of geodesics, parametrized by the interval \([0, \tau]\), joining the origin with the point \((x, y)\) is given by the number of solutions \( k \) of Equation (17).

We shall denote the right side term by \( \Psi_y : [0, 1] \to [0, 1] \)

\[
\Psi_y(k) = k \text{sn}(W^{-1}(y, k), k).
\]

Since \( |\text{sn} u| \leq 1 \), we have \( \lim_{k \to 0} \Psi_y(k) = 0 \).

Since \( \lim_{k \to 1} \text{sn}(u, k) = \tanh u \) and \( \lim_{k \to 1} \text{cn}(u, k) = \lim_{k \to 1} \text{dn}(u, k) = \text{sech} u \), then

\[
\rho(u) := \lim_{k \to 1} W(u, k) = u - \lim_{k \to 1} \int_0^u \text{dn}(s, k)^2 ds = u - \int_0^u \text{sech}^2 s ds = u - \tanh u.
\]

Since \( \rho'(u) = \tanh^2 u > 0 \) for \( u \neq 0 \) and

\[
\lim_{u \to \pm \infty} \rho(u) = \pm \infty,
\]

it follows that \( \rho \) is invertible with \( \rho^{-1} \) increasing. Hence

\[
\lim_{k \to 1} \Psi_y(k) = \tanh(\rho^{-1}(y)).
\]

Since \( \Psi_y(0+) = 0 \) and \( \Psi_y(1-) = \tanh (\rho^{-1}(y)) \), the intermediate value property of continuous functions yields the existence of at least one solution \( \in (0, 1) \) of Equation (17). More precisely we have the following result:

**Proposition 2.8** Consider the point \( P(x, y) \), with \( 0 < x \leq \pi/4 \) and \( y > 0 \).

If \( \sin x < \tanh (\rho^{-1}(y)) \), the origin and the point \( P \) can be joined by at least one geodesic. The number of geodesics is given by the number of solutions \( k \in (0, 1) \) of
the equation

\[ \sin x = \Psi_y(k). \]  \tag{18} 

In the following, we shall use the above proposition to recover the result of Section 2.1. Making \( x = 0 \) in Equation (18) yields

\[
\Psi_y(k) = 0 \iff \text{sn}(W^{-1}(y,k),k) = 0 \iff W^{-1}(y,k) = 2mK(k), \ m = 0, \pm 1, \pm 2, \ldots \\
y = W(2mK) \iff y = 2mK - E(2mK) \iff y = 2mK - 2mE \iff \\
y = 2m(K - E) \iff \frac{y}{2m} = K(k) - E(k),
\]

which is Equation (12). There is a solution \( k_m \) associated with each integer \( m \), which corresponds to a geodesic. Hence in the case \( x = 0 \) the number of geodesics is infinite.

Due to the periodicity of \( \sin x \) the results can be extended to any strip around \( x = 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots \). Points situated in distinct strips cannot be joined by a smooth geodesic.

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