STOCHASTIC OPTIMIZATION APPROACH OF CAR VALUE DEPRECIATION *

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Abstract. This paper presents a stochastic model for a car's value and its depreciation under random repairs modeled by a Poisson process; the usage functional is defined and the optimal selling time is estimated. Exact or approximative formulas are provided where possible. The car's value is evaluated as an asset with negative return and paying random normally distributed dividends at stochastic times which are Erlang distributed.

Key words. Poisson process, selling time, depreciation, variational problem

AMS subject classifications. 62P30, 60G99

1. Introduction and Motivation. This paper deals with often asked questions such as *When it is optimal to sell my car?*, or *Do I regret now that I bought a second-hand car instead of a new one?* Approaching these questions leads to certain optimization problems which car dealers might find attractive to solve in order to optimize their businesses.

Keeping the car for a long time is not optimal since the cumulated cost of repairs will soon worth more than a new car. Selling the car just shortly after buying it is not an optimal decision either because the value of the car depreciates the most at the beginning.

We shall become more precise about the conditions of this problem. Assume one buys a second-hand car which does not have a full coverage, so if it breaks down, the owner is solely responsible for the repairs. Therefore, the car owner would like to optimize the following two opposite effects:

1. Maximize the usage of the car, keeping the car for as long as possible;
2. Minimize the incidental costs, by paying as little as possible in repairs.

The value of the car goes down over time, while the cumulated value of repairs increases. It must be a time T in the future when the owner will decide that he is better off selling the car rather than keeping it and continuing to pour money into it. One of the proposed tasks of the present article is to describe the optimal selling time T. We shall treat this problem by setting up a stochastic differential equation for the value of the car at time t, which takes into consideration the depreciation of the car and the stochastic payments.

Sections 2 and 3 introduce the main hypotheses and notions needed to construct the stochastic models of later sections.

In section 4 we deal with a stochastic differential equation for the car's value and with its solution. The time when the car becomes valueless is also computed.

Section 5 is dedicated to defining and solving a variational problem defined by introducing the usage functional of a car as the expectation of the difference between the benefit and the loss encountered by keeping and using the car until time t.

Section 6 extends some results of previous sections in the case when the depreciation rate is stochastic. The reason behind this approach is the fact that the blue-book

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value of all cars made in the same year depends on the mileage, which itself is a random variable.

Another goal of the paper is to deal with the problem of comparing the following two deals:

1. Buying a new car;
2. Buying a second-hand car of a different brand and of the same price as the new car.

In order to characterize the better deal, we exploit the relations between average repair rates and depreciation rates of the two cars, see section 7.
Finally, in section 8 we discuss the conclusions and remarks regarding the models discussed in the paper.

The financial equivalent idea of this paper is that the value of a car behaves as the price of an asset with negative return (due to depreciation) that pays dividends of random magnitudes (the repair payments) at stochastic time instances. The idea of using jump-diffusion models have been studied in the academic finance community starting with Merton's seminal paper [3] and continued until today (see [4] and the references therein). This is a valuable tool recently accepted by banks in their daily market modeling when concerned about market shocks.

2. Basic notions. In this section we shall introduce and briefly describe the concepts and notations used in the following sections.

2.1. The blue-book value of a car. This concept refers to the market value of a car, which is provided by the largest USA automotive vehicle valuation company, Kelley Blue Book. The market value of any car depreciates over time. This process can be a complicated one, depending on several parameters, such as year, mileage, car's condition, etc. For the sake of simplicity we shall assume that the value of a car decreases over time according to the exponential law $V(t) = V_0 e^{-kt}$, where V_0 is the price of the car when it is new and k is the depreciation rate, which depends on the car's brand. We shall refer to $V(t)$ as the blue-book value of the car at time t. Section 6 extends this notion to the case when the depreciation rate k is stochastic, following a diffusion model. For the time being we shall assume that k is a positive constant.

2.2. The intrinsic value. This is the profit made by the owner when selling the car. This is obtained by subtracting the present value of all repair payments from the blue-book price at the selling time.

Assume a car is bought at time t_0 and sold at time t, both transactions being made at the blue-book values. Suppose there are n repair payments in values of $R_1, \dots, R_n$ made at time instances $t_1, \dots, t_n \in (t_0, t)$. These payments can't be recovered and hence they are subtracted from the selling price. Then the intrinsic value at time t, when starting from t_0, is given by

$$I(t_0, t) = V(t_0) e^{-k(t-t_0)} - \sum_{t_0 \le t_j \le t} R_j e^{r(t-t_j)}. \tag{2.1}$$

If $t_0 = 0$, we shall write $I(t) = I(0, t)$. The intrinsic value of the car behaves like the value of an asset that depreciates in time and has downside jumps from time to time. At a certain time the asset will have no value, and further will become negative, see Fig. 2.1.

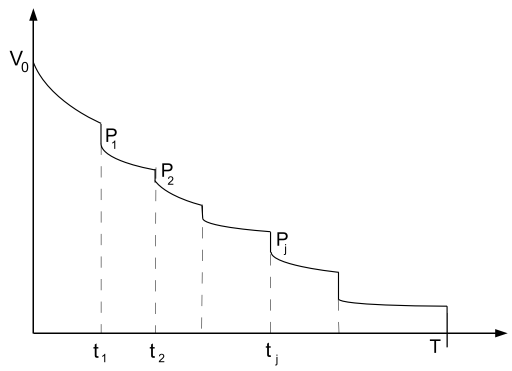

FIG. 2.1. *The intrinsic value of the car decreases to zero.*

2.3. The optimal selling time. We noticed that for t large the intrinsic value (2.1) becomes negative. The owner needs to sell the car at the latest time possible such that the intrinsic value is still positive. Formally, the optimal selling time T is defined by

$$T = \sup\{t > t_0; I(t_0, t) > 0\}.$$

After this time the intrinsic value of the car becomes negative i.e., $I(t_0, t) < 0$ for $t > T$.

If the instances t_j and the payments R_j are given, then the optimal time T can be found by solving the equation

$$V(t_0)e^{-k(T-t_0)} = \sum_{t_0 \le t_j \le T} R_j e^{r(T-t_j)} \tag{2.2}$$

by a trial and error procedure.

2.4. The cumulative repair payments. The right side of (2.2) represents the value at time T of all repair payments made in the interval $(t_0, T]$. The payments are taken by considering the time value of money at the bank risk-free rate r.

3. Basic Results. The results of this section will be used in the sequel.

3.1. Working hypotheses. Unfortunately, in a real life situation both time instances t_j and repair payments R_j are random variables, and the aforementioned easy way of finding the optimal selling time T does not work. In fact, the time T itself is a random variable and the deterministic value of concern will be the expected selling time $E[T]$.

Taking the randomness character of the problem into account, we can make the following acceptable assumptions:

1. The average of repairs per year is a known function (depending on time and car model); for the time being this is assumed constant;
2. The probability that the car needs to be fixed once in a time interval is proportional to the size of the interval;

3. The event that the car breaks down in an interval of time is independent of any other break down outside of that time interval;
4. The probability that more than one fix is needed in a short time interval is small;
5. The payments R_j are normally distributed random variables with the same mean and variance.

These assumptions lead to a model using a Poisson process. In the following N_t will denote the number of failures (and hence repairs) of the car until time t. N_t is a Poisson process with constant rate λ. For more properties of Poisson processes, the reader can consult [5].

3.2. Expected value of future value payments. Now, we shall accommodate the repair payments to our model.

Taking the value of money into account, the cumulative payment until time t, is given by the random variable

$$R(t) = \sum_{j=1}^{N_t} R_j e^{r(t-t_j)}.$$

It is worth noting that the payments R_i are independent of the number of repairs, N_t, and of repair times, t_i. It makes sense to consider a model in which the payments R_i are independent and normally distributed with mean $E[R_i] = \rho$. Since

$$\begin{aligned} E[R(t)|N_t = n] &= \rho e^{rt} E\big[\sum_{j=1}^{n} e^{-rt_j} | N_t = n\big] = \rho e^{rt} \frac{n}{t} \int_0^t e^{-rx}\, dx \\ &= \frac{n\rho}{rt}(e^{rt} - 1), \end{aligned}$$

using $E[R(t)] = E[E[R(t)|N_t]]$, yields that the expected value of future value of the total payments R_i until time t is given by

$$E[R(t)] = \frac{\lambda\rho}{r}(e^{rt} - 1). \tag{3.1}$$

More details on the previous computation can be found in [6], p. 71-72.

3.3. Selling time. Assume the car is sold at time t for its blue-book value, $V_0 e^{-kt}$, where k is the depreciation rate. If this value exceeds the cumulative repair costs until time t, denoted by $R(t)$, then the intrinsic value of the car is given by the difference

$$I(t) = V_0 e^{-kt} - R(t).$$

The selling time, which is defined as

$$T = \inf\{t; E[I_t] > 0\},$$

can be obtained in an equivalent way from $E[I_T] = 0$. Using (3.1) the selling time T satisfies the equation

$$V_0 e^{-kT} = \frac{\lambda\rho}{r}(e^{rT} - 1). \tag{3.2}$$

Since the left side is decreasing, while the right side is increasing in T, there is a unique solution $T > 0$ for the equation (3.2). An exact solution cannot be provided. However, replacing both terms by their linear approximations, we obtain the following rough approximation for the solution

$$T \approx \frac{V_0}{\lambda\rho + kV_0}. \tag{3.3}$$

Due to convexity reasons, formula (3.3) provides in fact a lower bound for the selling time T. If $r = 0$, the equation for the selling time T becomes

$$V_0 e^{-kT} = \lambda\rho T.$$

In its linear approximation this takes the form

$$V_0(1 - kT) = \lambda\rho T,$$

with the solution $T \approx V_0/(\lambda\rho + kV_0)$.

3.4. The time of the nth repair. If t_n denotes the time of the nth repair, then

$$\begin{aligned} P(t < t_n < t + dt) &= P(N_t = n-1)P\big(1 \text{ repair in } \ (t, t+dt)\big) \\ &= \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt, \end{aligned}$$

so the density function of the random variable t_n is

$$f(t) = \lambda \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}, \qquad t \geq 0,$$

which shows that t_n has an Erlang distribution with parameters n and $\beta = 1/\lambda$. In this case one waits on average for the nth repair to occur a time equal to n/λ. It is also known that the time intervals between two consecutive car failures, $t_n - t_{n-1}$, are independent random variables that are exponentially distributed with expectation $1/\lambda$, see for instance [5].

3.5. Payment duration. If each payment R_i is counted with the weight t_i, the random variable

$$D(t) = \sum_{i=1}^{N_t} t_i R_i$$

measures the duration of payments until time t. The duration will be used to compare a new and a second hand deal in section 7, see Corollary 7.2. The economic meaning of duration is how long one has to wait to get the payments.

This can be seen more clearly in the following example. Assume, for instance, the case of two annuities: an increasing annuity $R_1 < R_2 < \ldots < R_n$ and a decreasing annuity $\overline{R}_1 > \overline{R}_2 > \cdots > \overline{R}_n$, with payments R_i and $\overline{R}_i$ made at time t_i, $i = 1, \cdots, n$. Assume also that the total payments of both annuities are the same, i.e., $\sum_{i=1}^n R_i = \sum_{i=1}^n \overline{R}_i$. It can be shown as an application of Tchebychev's inequality (see section 2.17 of [10]) that

$$\sum_{i=1}^n t_i R_i > \frac{1}{n}\sum_{i=1}^n R_i \sum_{i=1}^n t_i > \sum_{i=1}^n t_i \overline{R}_i,$$

i.e. the duration of an increasing annuity is larger than the duration of a decreasing annuity.

We shall show that under previous hypotheses, the expected duration is quadratic over time. This will be computed using the following weighted average

$$E[D(t)] = \sum_{n=1}^{\infty} E[D(t)|N_t = n]P(N_t = n).$$

The conditional expectation component is given by

$$\begin{aligned} E[D(t)|N_t = n] &= E\Big[\sum_{i=1}^{N_t} t_i R_i | N_t = n\Big] = \sum_{i=1}^{n} E[R_i]E[t_i|N_t = n] \\ &= \rho E\Big[\sum_{i=1}^{n} t_i | N_t = n\Big]. \end{aligned}$$

Given the condition $N_t = n$, the arrival times t_i are distributed the same as the order statistics of n independent random variables, U_i, uniformly distributed on the interval $(0, t)$, see [6], p. 67. Then

$$E\Big[\sum_{i=1}^{n} t_i | N_t = n\Big] = E\Big[\sum_{i=1}^{n} U_i\Big] = \sum_{i=1}^{n} E[U_i] = nE[U_1] = n\frac{t}{2}.$$

Therefore

$$E[D(t)|N_t = n] = n\rho\frac{t}{2},$$

and hence

$$\begin{aligned} E[D(t)] &= \rho\frac{t}{2}\sum_{n=1}^{\infty} nP(N_t = n) \\ &= \rho\frac{t}{2}E[N_t] = \lambda\rho\frac{t^2}{2}. \end{aligned}$$

Hence the expected value of the duration of payments until time t is quadratic over time.

The formula of $E[D(t)]$ can be also obtained by stochastic integration as in the following. Since

$$dD(u) = D(u + du) - D(u) = \sum_{i=N_u}^{N_{u+du}} t_i R_i = \begin{cases} uR_u, & \text{with probability } \lambda du \\ 0, & \text{with probability } 1 - \lambda du, \end{cases}$$

using that $D(0) = 0$, integrating and taking the expectation yields

$$\begin{aligned} E[D(t)] &= \int_0^t E\big[\sum_{i=N_u}^{N_{u+du}} t_i R_i\big] = \int_0^t \lambda E[P_u]u\, du \\ &= \lambda\rho\frac{t^2}{2}. \end{aligned}$$

Two deals have the same expected duration if $E[D_1(t)] = E[D_2(t)]$ for some $t > 0$. Using the previous formula this is equivalent to $\lambda_1\rho_1 - \lambda_2\rho_2 = 0$.

4. A Stochastic Equation for the Car's Value. In this section we shall model the value of the car using a stochastic differential equation with jumps; solving it, we shall find the time T, when the car has no value. The interested reader can find more about stochastic differential equations in [8] and [9]. The value of the car is maximum at the initial time $t = 0$, and is equal to V_0. This value depreciates in time exponentially, with a rate depending on the car's brand. We consider that the car needs repairs that are modeled by a Poisson process, N_t, with constant rate λ. The cost of these repairs are considered independent and normally distributed with mean ρ. They are also considered independent with respect to the number of failures N_t. These repairs are needed at times $t_1, t_2, \cdots$, which are gamma distributed. If at time t_1 the first repair occurs, then the value of the car decreases by the amount of the repair cost R_1; the jump discontinuity at t_1 can be written as

$$V_{t_1-} = V_0 e^{-kt_1}, \qquad V_{t_1+} = V_0 e^{-kt_1} - R_1.$$

Before setting up the stochastic differential equation, consider the following notations

- V_t = the value of the car at time t; this means the value invested in the car until time t;[1]
- k = the depreciation rate;
- R_t = the payment for a repair at time t;
- ρ = the average cost of a repair;
- N_t = the Poisson process with rate λ.

4.1. Constant Payments. We shall make the assumption in the beginning that all repair costs are equal to ρ. In this model the car's value decreases due to the following two independent influences:

- The value V_t depreciates exponentially over time; in the absence of repairs, V_t would satisfy the equation
$$dV_t = -kV_t dt,$$
with the solution $V_t = V_0 e^{-kt}$, where V_0 stands for the initial value of the car. This coincides with the blue-book value $V(t)$.
- Each time a repair occurs, the car's value jumps down by a constant amount ρ, the cost of the repair. After we incorporate the jumps, the value V_t satisfies the following stochastic differential equation
$$dV_t = -kV_t dt - \rho dN_t. \tag{4.1}$$

It is worth noting that the unpredictable part ρdN_t does not depend on the car's value V_t. Multiplying by the integrating factor e^{kt}, the equation transforms into an exact equation

$$d(e^{kt}V_t) = -\rho e^{kt} dN_t.$$

Integrating between 0 and t, and solving for V_t, yields

$$V_t = e^{-kt}V_0 - \rho e^{-kt} \int_0^t e^{ks}\, dN_s.$$

Since the first term on the right side tends to zero as $t \to \infty$, and the second one is non-positive, it follows that the expected value $E[V_t]$ becomes negative for t large

[1] This is not the blue-book value, which was denoted by $V(t)$.

enough. It must be a time T, such that $E[V_T] = 0$, i.e. the car's value is expected to be worthless. Then the selling time should be at most T. The equation satisfied by T can be also written in the form

$$E\Big[\int_0^T \rho e^{ks}\, dN_s\Big] = V_0. \tag{4.2}$$

In the following we compute the value of the expectation on the left side:

$$\begin{aligned}\int_0^T \rho e^{ks}\, dN_s &= \int_0^T \rho e^{ks}(dM_s + \lambda ds)\\ &= \int_0^T \rho e^{ks}\, dM_s + \lambda \int_0^T \rho e^{ks}\, ds\\ &= \rho \int_0^T e^{ks}\, dM_s + \frac{\lambda\rho}{k}(e^{kT} - 1),\end{aligned}$$

where $M_t = N_t - \lambda t$ is the compensated Poisson process. Since the integral with respect to dM_t is a martingale, taking the expectation yields

$$E\Big[\int_0^T \rho e^{ks}\, dN_s\Big] = \frac{\lambda\rho}{k}(e^{kT} - 1).$$

It is worth noting the similarity between this formula and the one given by (3.1). Then the equation (4.2) becomes

$$\frac{\lambda\rho}{k}(e^{kT} - 1) = V_0,$$

with the solution

$$T = \frac{1}{k}\ln\Big(1 + \frac{kV_0}{\lambda\rho}\Big). \tag{4.3}$$

At time T we can give up the car for free, since it is valueless. If we use the linear approximation $\ln(1 + x) \approx x$, as $x \to 0$, we obtain that for k small (negligible depreciation rate) we have the simple formula

$$T = \frac{V_0}{\lambda\rho}. \tag{4.4}$$

We can arrive to the same formula if let $k = 0$ in the equation (4.1)

$$dV_t = -\rho dN_t.$$

Integrating yields $V_t = V_0 - \rho N_t$, so $E[V_t] = V_0 - \rho E[N_t] = V_0 - \rho\lambda t$. This vanishes for $t = \dfrac{V_0}{\lambda\rho}$.

4.2. Stochastic Payments. The formula for T obtained in the preceding subsection holds even in the general case when the payments are given by a stochastic process R_t with $E[R_t] = \rho$. The car's value V_t satisfies the stochastic differential equation

$$dV_t = -kV_t dt - R_t dN_t.$$

Solving as an exact equation yields $d(e^{kt}V_t) = -R_t e^{kt} dN_t$, which after integration leads to the solution

$$V_t = V_0 e^{-kt} - \int_0^t R_s e^{k(s-t)}\, dN_s.$$

The previous integral with respect to the Poisson process N_t is taken in the Stieltjes sense.

We are looking now for a time T that satisfies $E[V_T] = 0$. At this time the expected value of the car vanishes. This is equivalent with

$$E\Big[\int_0^T R_s e^{ks}\, dN_s\Big] = V_0.$$

Reducing the integral to a martingale by using the compensated Poisson process $M_t = N_t - \lambda t$ yields

$$E\Big[\int_0^T R_s e^{ks}\, dM_s + \lambda \int_0^T R_s e^{ks}\, ds\Big] = V_0,$$

which becomes

$$E\Big[\lambda \int_0^T R_s e^{ks}\, ds\Big] = V_0 \Longleftrightarrow \lambda \int_0^T E[R_s] e^{ks}\, ds = V_0 \tag{4.5}$$

from where

$$\frac{\lambda\rho(e^{kT} - 1)}{k} = V_0,$$

which has the solution $T = \frac{1}{k}\ln\Big(1 + \frac{kV_0}{\lambda\rho}\Big)$. We note the similarity with formula (4.3).

4.3. Payments following a stochastic process. Neither the assumption that the payments are equal, nor that the payments have all the same mean is too realistic. In real life, the first repair costs are in the beginning zero, after which they start increasing in price. One model that satisfies this requirement assumes the repair payments R_t satisfying the stochastic differential equation

$$dR_t = \alpha(1 + R_t)dt + \sigma R_t dW_t, \qquad R_0 = 0.$$

The constant α is positive and measures the rate at which the payment increases; the term $\sigma R_t dW_t$ models the unpredictable part. The constant σ is denoting the volatility of payments. It is worth noting that the unpredictable term is proportional with the payment R_t. Integrating and taking the expectation yields

$$E[R_t] = \int_0^t \alpha(1 + E[R_s])\, ds + E\Big[\int_0^t \sigma R_s\, dW_s\Big] = \int_0^t \alpha(1 + E[R_s])\, ds.$$

Differentiating we get

$$\frac{d}{dt}E[R_t] = \alpha(1 + E[R_t]), \qquad E[R_0] = 0.$$

Solving yields

$$E[R_t] = e^{\alpha t} - 1.$$

Substituting in (4.5) we get

$$\lambda \int_0^T (e^{\alpha t} - 1) e^{kt} \, dt = V_0.$$

This equation has a unique solution T, which can be obtained as the solution of the equation

$$\frac{e^{(\alpha+k)T} - 1}{\alpha + k} - \frac{e^{kT} - 1}{k} = \frac{V_0}{\lambda}.$$

No closed form solution can be found for this exponential equation. However, replacing the exponentials by their quadratic approximations yields a quadratic equation with the solution

$$T = \sqrt{\frac{2V_0}{\alpha\lambda}}.$$

5. The Variational Problem. We shall associate a variational problem that optimizes the usage function of a car and find time t^* when the usage is maximum. In order to do this we take into account the loss and the gain until time t.
The loss $L(t)$ encountered until time t is due to two contributions:

1. Total cumulative losses due to repair payments until time t, given by

$$R(t) = \sum_{i=1}^{N_t} R_j e^{r(t-t_j)}.$$

2. Loss due to depreciation. Consider the time value of money and the depreciation until time t, the value of the loss is given by $V_0 e^{(r-k)t}$. The car appreciates in time as an investment with interest rate r, and depreciates simultaneously at rate k. In general, the depreciation rate k is much larger than the interest rate r. There might be some cases, such as that of a 1960 Buick that is kept as an investment, where the inequality is reversed.

The gain or benefit of usage until time t is proportional to the intensity of usage (how many miles are put on the car) and time duration of usage. The easiest model for the benefit function is the linear one

$$B(t) = \Gamma t,$$

where Γ is the intensity of usage. This says that the more intense and longer the car is used, the larger the benefit. Since Γ is constant, it means that the car is uniformly used. More elaborate models might consider that $\Gamma = \Gamma(t)$ is a function of time. For instance, one might prefer to use the new car less than his older car. In this general situation we may consider the benefit in the integral form

$$B(t) = \int_0^t \gamma(s) \, ds,$$

where $\gamma(t)$ stands for the density of intensity of usage.

We shall consider the following case when Γ is constant and make a few remarks about the general case later. If the car is used with more intensity, the odds of breaking down are higher; so if Γ increases, then the rate λ increases; this means that Γ is an increasing function of λ. Since cars that are never used do not need to be repaired, then for $\lambda = 0$, we have $\Gamma = 0$. One model that satisfies this condition is the linear model $\Gamma = \beta\lambda$, with $\beta > 0$ constant. This model is not realistic, since it implies that if a car is driven twice as much as another car, then it must need twice as much repairs too. This is not true, and a better model that fits reality is the exponential relation $\Gamma = e^{\beta\lambda} - 1$. This means that when the car usage increases exponentially, the rate of repairs increases linearly.

The usage function is defined as the difference between the gain and the loss until time t

$$U(t) = B(t) - L(t) = \Gamma t - V_0 e^{(r-k)t} - R(t).$$

Consider the following optimization problem

$$\max_{t \geq 0} E[U(t)].$$

The optimal solution, t^*, is the time when the expected usage function reaches its maximum. The solution might not be unique, or exist at all; we shall discuss this issue at the end of the section. Consider the function

$$\begin{aligned} F(t) &= E[U(t)] \\ &= \Gamma t - V_0 e^{(r-k)t} - E[R(t)] \\ &= \Gamma t - V_0 e^{(r-k)t} - \frac{\lambda\rho}{r}(e^{rt} - 1) \end{aligned}$$

with derivatives

$$\begin{aligned} F'(t) &= \Gamma - (r-k)V_0 e^{(r-k)t} - \lambda\rho e^{rt} \\ F''(t) &= -(r-k)^2 V_0 e^{(r-k)t} < 0. \end{aligned}$$

From the second derivative test, the solution t^* of the equation

$$\Gamma e^{-rt} = \lambda\rho + (r-k)V_0 e^{-kt} \tag{5.1}$$

is a maximum point for $F(t)$. The number of solutions of equation (5.1) depends on the values of the coefficients Γ, ρ, r, k and V_0.

In normal market conditions, the depreciation rate exceeds the interest rate, $k > r$. The next result shows that under this hypothesis the aforementioned variational problem has at most one solution.

PROPOSITION 5.1. *Let $k > r$.*
(i) If $\Gamma \geq \lambda\rho - (k-r)V_0$, then the equation (5.1) has a unique solution t^*.
(ii) If $\Gamma < \lambda\rho - (k-r)V_0$, then the equation (5.1) has no solutions; this means that if the car is used too little, we can't get the maximum usage out of it.

Proof. (*i*) Let $\phi(t) = \Gamma e^{-rt}$ and $\xi(t) = \lambda\rho - (k-r)V_0 e^{-kt}$ be the left and the right side of (5.1), respectively. The function $\phi(t)$ is decreasing and $\xi(t)$ is increasing, with

$$\begin{aligned} \Gamma = \phi(0) &\geq \xi(0) = \lambda\rho - (k-r)V_0 \\ 0 = \phi(\infty) &< \xi(\infty) = \lambda\rho. \end{aligned}$$

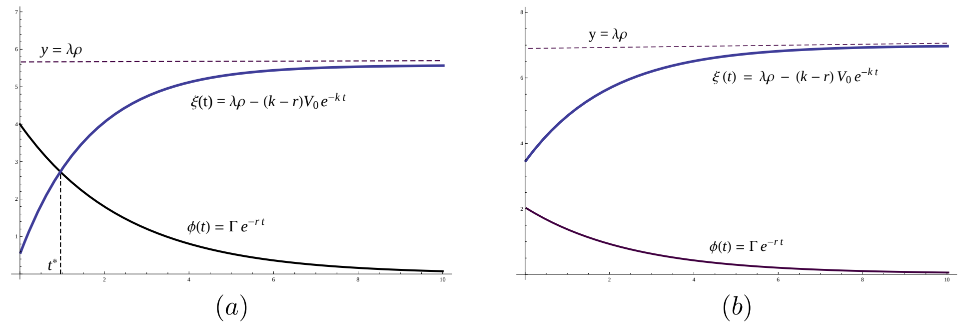

FIG. 5.1. (a) *Unique optimal solution* t^* *for* $\Gamma > \lambda\rho - (k-r)V_0$*;* (b) *No optimal solutions if* $\Gamma < \lambda\rho - (k-r)V_0$*.*

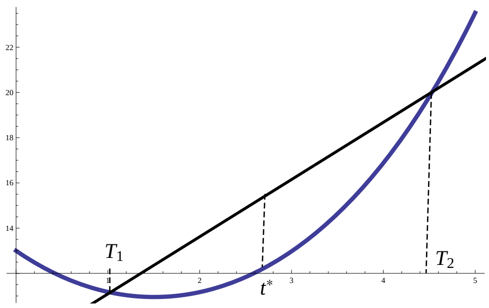

FIG. 5.2. *The optimal time* t^* *which maximizes the usage function.*

From continuity reasons, the equation (5.1) has a unique solution t^*, and hence the variational problem has only one optimal solution, see Fig. 5.1(a).

(ii) From the monotonicity of $\phi(t)$ and $\xi(t)$ and the inequality

$$\phi(t) \leq \phi(0) = \Gamma < \lambda\rho - (k-r)V_0 = \xi(0) \leq \xi(t),$$

it follows that the equation (5.1) has no solutions, see Fig. 5.1(b). $\square$

The optimal usage time t^* cannot be obtained in closed form; however, we can compute a lower bound for t^*. Due to convexity, the tangent lines to the graphs of functions $\phi(t)$ and $\xi(t)$ at points $\big(0, \phi(0)\big)$ and $\big(0, \xi(0)\big)$ intersect at a point with coordinate t_1, with $t_1 < t^*$. The value t_1 is the solution of the linear equation

$$\Gamma - r\Gamma t = \lambda\rho - (k-r)V_0 + kV_0(k-r)t$$

and is given by

$$t_1 = \frac{\Gamma - \lambda\rho + (k-r)V_0}{r\Gamma + kV_0(k-r)}.$$

This is a lower bound for the optimal solution t^*. Selling the car before the optimal solution t^* is not a wise idea, since the maximum usage has not been reached yet. Hence the car must be kept strictly longer than t_1.

5.1. Selling Interval. The expected usage function $E[U(t)]$ is positive for $T_1 < t < T_2$ and negative otherwise, where T_1 and T_2 are the solutions of the equation

$$\Gamma t - \frac{\lambda\rho}{r} = e^{rt}\Big(\frac{\lambda\rho}{r} + V_0 e^{-t}\Big).$$

The left side is a linear function and the right side is a convex function with infinite limit as $t \to \infty$, see Fig. 5.2. The maximum difference occurs for $t = t^*$. Since it makes sense to keep the car at least until the maximum usage time t^* is reached, and not to keep it if the usage is negative, then the optimal selling time T_{sell} is somewhere in the interval (t^*, T_2). Since T_{sell} is a random variable, we cannot say that it is exactly T_2, but it makes sense to look for it somewhere shortly before T_2.

5.2. Uniqueness Condition for t^*. Proposition 5.1 assures that in the case $\Gamma \geq \lambda\rho - (k-r)V_0$ the optimal solution t^* is unique. We have assumed that the intensity of usage Γ is related to the rate of repairs λ by the relation $\Gamma = e^{\beta\lambda} - 1$. The goal of this subsection is to provide a value for β that provides uniqueness for t^*. The previous inequality can be written equivalently as

$$e^{\beta\lambda} \geq 1 + \lambda\rho - (k-r)V_0. \tag{5.2}$$

Expanding we have

$$1 + \beta\rho + \beta^2\rho^2/2 + \cdots \geq 1 + \lambda\rho - (k-r)V_0.$$

An obvious sufficient condition for this inequality to hold is $\beta \geq \rho$. So for any β larger than ρ there is a unique optimal time t^*.

6. Stochastic Depreciation Rate. Until now the depreciation rate k was considered constant; however, if the intensity of usage varies, then the depreciation also changes. In this section we shall let the depreciation rate k be stochastic.

We shall assume that the rate k_t is a random variable around a fixed value k, with a perturbation given by a Brownian motion W_t

$$k_t = k + \sigma W_t, \qquad t > 0,$$

where σ is a positive constant which controls the volatility of the process. In the absence of any repairs, the car's value satisfies the stochastic differential equation

$$dV_t = -(k + \sigma W_t)V_t dt,$$

with the solution

$$V_t = V_0 e^{-kt} e^{-\sigma \int_0^t W_s\, ds}.$$

This is a random variable with mean and variance

$$E[V_t] = V_0 e^{-kt + \frac{\sigma^2 t^3}{6}}$$
$$Var[V_t] = V_0^2 e^{-2kt} e^{\frac{\sigma^2 t^3}{3}} (e^{\frac{\sigma^2 t^3}{3}} - 1),$$

where we used that $\displaystyle\int_0^t W_s\, ds$ is normally distributed with mean 0 and variance $t^3/3$, see for instance [6]. If repairs are taken into consideration, the car's value satisfies the stochastic differential equation

$$dV_t = -(k + \sigma W_t)V_t dt - R_t dN_t.$$

Multiplying by the integrating factor $e^{-\int_0^t (k+\sigma W_s)\,ds}$ and integrating yields the solution

$$V_t = e^{-kt}e^{-\sigma\int_0^t W_s\,ds}V_0 - e^{-kt}e^{-\sigma\int_0^t W_s\,ds}\int_0^t R_s e^{ks}e^{\sigma\int_0^s W_u\,du}\,dN_s.$$

Since the rate k_t is independent of repairs and size of payments, it makes sense to consider W_t independent of R_t and N_t. Using independence and the martingale property, we can compute the expected value of V_t

$$\begin{aligned} E[V_t] &= e^{-kt}E[e^{-\sigma\int_0^t W_s\,ds}]V_0 - e^{-kt}E\Big[\int_0^t R_s e^{ks}e^{-\sigma\int_s^t W_u\,du}\,dN_s\Big] \\ &= e^{-kt}e^{\sigma^2 t^3/6}V_0 - E\Big[\int_0^t R_s e^{ks}e^{-\sigma\int_s^t W_u\,du}\,dM_s\Big] \\ &\quad -\lambda E\Big[\int_0^t R_s e^{ks}e^{-\sigma\int_s^t W_u\,du}\,ds\Big] \\ &= e^{-kt}e^{\sigma^2 t^3/6}V_0 - \lambda\int_0^t E[R_s]e^{ks}E\big[e^{-\sigma\int_s^t W_u\,du}\big]\,ds\Big] \\ &= e^{-kt+\frac{\sigma^2 t^3}{6}}V_0 - \frac{\lambda^2\rho(e^{rt}-1)}{r}\int_0^t e^{ks+\frac{\sigma^2(t-s)^3}{6}}\,ds. \qquad (6.1) \end{aligned}$$

The next result deals with the time when we expect the car to loose all its value.

PROPOSITION 6.1. *The equation $E[V_t]=0$ has at least one solution $t>0$.*

Proof. Making the substitution $u=t-s$ in the last integral of (6.1) we have that $E[V_t]=0$ is equivalent to

$$\begin{aligned} e^{-kt}e^{\frac{\sigma^2 t^3}{6}}V_0 &= \frac{\lambda^2\rho(e^{rt}-1)}{r}e^{kt}\int_0^t e^{-ku+\frac{\sigma^2 u^3}{6}}\,du \iff \\ \underbrace{e^{-kt}V_0}_{=f(t)} &= \underbrace{\frac{\lambda^2\rho(e^{rt}-1)}{r}e^{kt}\frac{\int_0^t e^{-ku+\frac{\sigma^2u^3}{6}}\,du}{e^{\frac{\sigma^2t^3}{6}}}}_{=g(t)}. \end{aligned}$$

We have

$$f(0)=V_0>g(0)=0, \qquad f(\infty)=0<g(\infty)=\infty,$$

since by L'Hospital's rule $g(t)\sim\frac{\lambda^2\rho(e^{rt}-1)}{r}$ as $t\to\infty$. By continuity reasons, the graphs of f and g intersect. Hence we have at least one solution $t>0$. ☐

An example where stochastic depreciation rates might appear is when for pricing reasons mileage is taken into consideration. Consider the model in which the market value of a car depends explicitly on two parameters: time t and mileage $m(t)$. The value at time t is supposed to be

$$V_t = V_0 e^{-kt}e^{-\mu m(t)},$$

where

$$k = -\frac{\partial V_t}{\partial t}, \qquad \mu = -\frac{\partial V_t}{\partial m(t)}$$

are the depreciation rates with respect to time and mileage, respectively. It makes sense to consider the mileage at time t to be given by the linear relation $m(t) = ct$, where c is the constant rate at which we put miles on the car. If the rate c is stochastic, with $c = c_0 + \sigma W_t$, then

$$V_t = V_0 e^{-kt} e^{-\mu(c_0+\sigma W_t)t} = V_0 e^{-k_t t},$$

with $k_t = k + c_0 + \sigma W_t$ stochastic rate.

7. Comparison between a new and a second-hand deal. In this section we present a comparison analysis between buying either a new car or a second-hand car of different brands at the same purchasing price. We assume that the second-hand car is of a better brand than the new one; therefore

- the car with the better brand depreciates slower than the new one; this means $k > \bar{k}$;
- the expected repair payments for the new car is less than the expected repair payments for the second-hand car; this means $\rho < \bar{\rho}$;
- the rate of repairs for the second-hand car exceeds the rate of repairs for the new car, $\bar{\lambda} > \lambda$.

We shall investigate the conditions under which there is a separation time t^* such that if $t < t^*$, then buying the second-hand car is a better deal; and if $t > t^*$, then buying the new car is a better deal. So buying the new car is a better deal in the long run, while buying the second-hand car is a better deal in the short run.

Let the purchasing prices of the new and second-hand cars be the same, $V_1 = \overline{V_1}$, and assume that the following inequality holds

$$E[V_1 e^{-kt} - R(t)] < E[\bar{V}_1 e^{-\bar{k}t} - \bar{R}(t)],$$

which means the expected value of the second-hand car is larger than the expected value of the new car. This can be written as

$$E[V_1 e^{-kt} - \bar{V}_1 e^{-\bar{k}t}] \le E[R(t) - \bar{R}(t)],$$

which becomes

$$\begin{aligned} V_1(e^{-kt} - e^{-\bar{k}t}) &\le E\Big[\sum_{j=1}^{N_t} R_j e^{r(t-t_j)} - \sum_{j=1}^{N_t} \bar{R}_j e^{r(t-t_j)}\Big] \\ &= \sum_{j=1}^{N_t} e^{r(t-t_j)} E[(R_j - \bar{R}_j)] \Longleftrightarrow \end{aligned}$$

$$V_1(e^{-kt} - e^{-\bar{k}t}) \le \frac{1}{r}(e^{rt} - 1)(\lambda\rho - \bar{\lambda}\bar{\rho}). \tag{7.1}$$

For which values of t is the inequality (7.1) satisfied? We shall answer this question in the following result.

Proposition 7.1.

(i) Let $\dfrac{\bar{\lambda}\bar{\rho} - \lambda\rho}{k - \bar{k}} < V_1$. *Then there is a separation time* t^* *such that the inequality* (7.1) *is satisfied for* $t < t^*$, *see Fig. 7.1(a); therefore it is a better deal to buy a second-hand car and keep it for a time less than* t^*; *if one is planning to keep the car longer than* t^*, *then the better deal would be to buy a new car.*

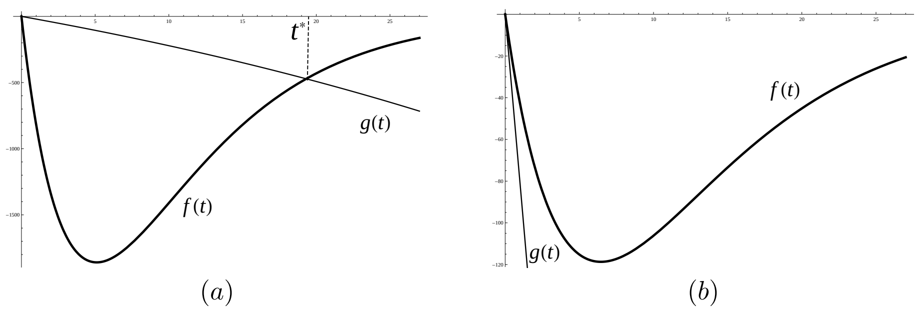

FIG. 7.1. *(a) The separation time t^*; (b) No separation time case: $f(t) > g(t)$.*

(ii) Let $\dfrac{\bar{\lambda}\bar{\rho} - \lambda\rho}{k - \bar{k}} \geq V_1$. *Then the better deal is always to buy a new car, see Fig. 7.1(b).*

Proof. Consider the functions

$$f(t) = V_1(e^{-kt} - e^{-\bar{k}t}), \quad g(t) = \frac{1}{r}(e^{rt} - 1)(\lambda\rho - \bar{\lambda}\bar{\rho}).$$

We have

$$f(0) = 0, \quad g(0) = 0, \quad f(+\infty) = 0, \quad g(+\infty) = -\infty$$

$$f'(0) = V_1(\bar{k} - k), \quad g'(0) = \lambda\rho - \bar{\lambda}\bar{\rho}.$$

We have two cases:

If $g'(0) > f'(0)$, then the graphs of f and g intersect by continuity reasons; so there is a separation time $t^* > 0$ such that the inequality (7.1) is satisfied for $t < t^*$, see Fig. 7.1(a). In this case buying a second-hand car is a better deal. This condition is equivalent to

$$\frac{\bar{\lambda}\bar{\rho} - \lambda\rho}{k - \bar{k}} < V_1.$$

If $g'(0) \leq f'(0)$, then the graph of f is above the graph of g, see Fig. 7.1(b). This condition is equivalent to

$$\frac{\bar{\lambda}\bar{\rho} - \lambda\rho}{k - \bar{k}} \geq V_1.$$

In this case the value of a new car will always exceed the value of a second-hand car. □

Let $\overline{D}(t)$ and $D(t)$ be the durations until time t of the aforementioned two deals.

COROLLARY 7.2. *If there is a time $t > 0$ such that*

$$E[\overline{D}(t) - D(t)] < \frac{1}{2}V_1(k - \bar{k})t^2$$

then there is a separation time t^* and it is a better deal to buy a second-hand car and keep it for a time less than t^*; if planning to keep the car longer than t^*, then the better deal is to buy a new car.

It is worth noting that if the deals have the same expected duration, then $\bar{\lambda}\bar{\rho} - \lambda\rho = 0$ and the conclusion of the previous corollary still holds.

8. Discussion and Conclusions. All the computations in this paper have been carried out under the hypothesis that the Poisson process N_t has a constant rate λ. In real life, the rate of repairs is not constant in time; it is always an increasing function of time, $\lambda(t)$, with the initial value $\lambda(0) = 0$. We can model this function using an exponential model of the type $\lambda(t) = a(e^{bt} - 1)$, with a and b constants. The Poisson process in this case is nonhomogeneous, with

$$P\big(N_b - N_a = k\big) = \frac{e^{-\lambda_{a,b}}(\lambda_{a,b})^k}{k!}, \qquad k = 0, 1, 2, \cdots,$$

where $\lambda_{a,b} = \displaystyle\int_a^b \lambda(t)\, dt$.

Under this assumption, the expected value of payments until time T takes the following integral form

$$E[R(T)] = \rho e^{rT} \int_0^T \lambda(t) e^{-rt}\, dt.$$

It is worth noting that $E[R(T)]$ is a strictly increasing function of T. For instance, if we assume the model $\lambda(t) = a(e^{bt} - 1)$, then

$$E[R(T)] = a\rho\Big(\frac{e^{bT} - e^{rT}}{b - r} + \frac{1 - e^{rT}}{r}\Big).$$

Even in this particular model, most of the closed form solutions obtained through the paper are not analytically trackable.

Some hypotheses of section 5 can be relaxed. The results of Proposition 5.1 do not change in an essential way if the benefit function is provided by a density function, $B(t) = \int_0^t \gamma(s)\, ds$. In this case equation (5.1) becomes

$$\gamma(t)e^{-rt} = \lambda\rho + (r - k)V_0 e^{-kt}. \tag{8.1}$$

If the density γ satisfies the following conditions

1. $\gamma \geq 0$
2. γ is decreasing
3. $\lim_{t\to\infty} \gamma(t)e^{-rt} = 0$
4. $\gamma(0) > \lambda\rho - (k - r)V_0$

then the equation (8.1) has a unique solution t^*. It is worth noting that a decreasing density functions means that the car usage decreases in time. This condition assures also the solution uniqueness.

It was shown that a sufficient condition for the uniqueness of the optimal time t^* is the inequality $\beta > \rho$. In fact sharper inequalities can be obtained. The smallest value of $\beta > 0$ for which the inequality (5.2) holds can be pursued by investigating a problem of the type $\beta > \max \frac{\ln(mx+b)}{x}$.

More elaborate models for the stochastic depreciation can be considered in section (6). For instance, one may assume that the rate k_t satisfies a mean reverting Ornstein process, but the computations in this care are by far much more complex.

In section 7 we arrive at the conclusion that buying a new car is always a better deal if planning to keep the car long enough. However, under hypothesis (i), in short run it would be more beneficial to buy a second-hand car. There is no closed form solution for the separation time t^*; however numerical methods can be used to find t^* without difficulty.

As specified before, the financial equivalent of the car's value is a dividend-paying asset with a negative return. In finance the time and size of dividends are usually known from the beginning of the contract, while here both are random variables with given distributions. Among similar problems with finance, the authors plan to address in their future work we specify:

One buys a car from a dealer. How much premium he should pay to gain the right to sell the car back to the dealer after a given time T for the price K, provided the value value at time T is $V_T < K$, with K chosen by the dealer at the beginning of the contract?

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