Global connectivity and optimization on an infinite step distribution

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Abstract

This paper provides an example of a 2-dimensional distribution that does not satisfy Chow’s bracket generating condition, but is horizontally connected. We prove the global connectivity by non-holonomic geodesics. Explicit solutions for geodesics are obtained. © 2012 Elsevier Inc. All rights reserved.

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0. Introduction

A manifold with constraints is a pair \((M, \mathcal{H})\), where \(M\) is a finite dimensional connected manifold and \(\mathcal{H}\) is a distribution, i.e. a smooth assignment \(x \rightarrow \mathcal{H}_x \subseteq T_x M\), where \(T_x M\) denotes the tangent space of \(M\) at \(x\). The horizontal distribution \(\mathcal{H}\) is usually given as the linear hull of a set \(D = \{X_1, X_2, \ldots, X_m\}\) of smooth vector fields on \(M\). The vector fields can be viewed as the dual of \(m\) one-forms \(\{\omega_1, \ldots, \omega_m\}\) that are equivalent with the system's constraint conditions.

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Many problems from mechanics can be described by parameters that belong to a manifold with constraints [1,4]. This way we have control over the velocities of the system, which belong to the non-integrable distribution $\mathcal{H}$ defined by some non-holonomic constraints. One of the main problems is to determine whether there is a continuous curve tangent to the distribution $\mathcal{H}$ and connecting any two given points of the manifold $M$.

It was shown by Chow and Rashevskii [5,8] that if the bracket generating condition holds, then the system can move continuously, piece-wise differentiably, between any two given states. The bracket generating condition says that the vector fields which generate the distribution, together with finitely many of their iterated brackets span the tangent space of the coordinate space at each point; this means that for each $x \in M$, there is an $r > 1$ such that

$$X_i, \ldots, [X_i, X_j], \ldots, [X_i, [X_j, X_k]], \ldots, \cdots [X_i, \ldots, [X_i, X_{i+1}] \cdots]$$

span $T_x M$.

This condition appears also in Hörmander [6] as a sufficient, but not necessary condition for a differential operator defined as a sum of squares of vector fields to be hypoelliptic. For example, consider the vector fields on $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} + \alpha e^{-(x^2+y^2)^{-\alpha}} x(x^2+y^2)^{-2(\alpha+1)} \frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} + \alpha e^{-(x^2+y^2)^{-\alpha}} y(x^2+y^2)^{-2(\alpha+1)} \frac{\partial}{\partial t},$$

where $\alpha$ is a positive number. It is easy to see that $D = \{X, Y\}$ is a horizontal subbundle of the hypersurface $\{(z_1, z_2) \in \mathbb{C}^2: \text{Im}(z_2) = \exp[-|z_1|^{-2\alpha}])$ with $z_1 = x + iy$ and $\text{Re}(z_2) = t$. However, the above vector fields do not satisfy Chow’s condition at the origin. But, it can be proved that the sub-Laplacian $X^2 + Y^2$ is hypoelliptic if $\alpha$ is sufficiently small. We shall discuss this problem in a forthcoming paper (see [3]).

If the number of brackets needed to generate all the missing directions at a point $x$ is denoted by $k$, then it is said that the distribution is of step $k + 1$ at the point $x$. Therefore, the step 1 corresponds to the case when the distribution is the entire tangent bundle of a Riemannian manifold.

To make our concepts more precise, we consider the following definition:

**Definition 1.** The manifold with constraints $(M, \mathcal{H})$ is horizontally connected if for any two points $A, B \in M$, there is a piece-wise smooth curve that is tangent to the distribution $\mathcal{H}$ joining the points $A$ and $B$.

When the bracket generating condition holds, Chow’s theorem implies that the manifold with constraints $(M, \mathcal{H})$ is horizontally connected. The converse of Chow’s theorem, that is, if the manifold with constraints is horizontally connected then the bracket generating condition holds, is in general, not true.

Let $\gamma^X_t(x)$ denote the integral curve of $X$ that passes through $x$ when $t = 0$. In the next result “reachability” means horizontal connectivity, and $P_D$ denotes the smallest distribution $\mathcal{H}$ with the following properties:

(i) every element of $D$ belongs to $\mathcal{H}$;

(ii) $\mathcal{H}$ is $D$-invariant, i.e. if $x \in M$, $X \in D$, and $t \in \mathbb{R}$ such that $\gamma^X_t(x)$ is defined, it follows that the differential of $\gamma^X_t(x)$ maps $\mathcal{H}_x$ into $\mathcal{H}_{\gamma^X(t)}$.

A necessary and sufficient condition for connectivity by horizontal curves is given by Theorem 7.1 of [9]:

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The Horizontal Curves Connectivity Condition. Let $D$ be an everywhere defined set of $C^\infty$ vector fields on the $n$-dimensional $C^\infty$ manifold $M$. Then $D$ satisfies the reachability condition if and only if $P_D(x)$ has dimension $n$ for every $x \in M$.

It is worth noting that the distribution $P_D$ is involutive and its maximal integral manifolds are the orbits of $D$ (see Theorem 2 of [10]), where by the orbit of a point $x \in M$ we mean the set of points in $M$ that can be joined with $x$ by a piece-wise smooth horizontal curve.

Let $D^*$ denote the smallest set of vector fields on $M$ that contains the set $D = \{X_1, X_2, \ldots, X_m\}$ and is closed with respect to Lie brackets. Let $H^*$ denote the distribution spanned by $D^*$. Then a useful relation contained in [9] is

$$H \subseteq H^* \subseteq P_D.$$ (0.1)

Then Chow’s theorem can be reformulated in the following equivalent form:

If the distribution $H^*$ has the maximal dimension at each point, then $(M, H)$ is horizontally connected.

The fact that $H^*$ has maximal dimension implies that $P_D$ has the same property, since $H^*_x = P_D(x) = T_x M$, so Sussmann’s theorem implies Chow’s theorem. However, the second inclusion of (0.1) might be proper sometimes, $H^* \neq P_D$.

In this paper we present an example of manifold with constraints that does not satisfy the bracket generating condition but is horizontally connected; this means $H^* \neq P_D$ and $P_D$ is of maximal dimension. Another example of this type can be found in [9, p. 185]. However, our example is more natural and comes from an optimization parking problem of a two-wheel cart, see Fig. 1(a).

The plan of the paper is as follows. In Section 1 we recall some non-holonomic geometry of the rolling disk. In the second section we construct the non-holonomic geometry of a two-wheel cart which leads to our counter-example. In Section 3 we solve the Euler–Lagrange equations and in Section 4 we prove the global connectivity by horizontal curves.

1. The rolling disk

This section recalls the rolling disk, a simple and widely studied classical model of manifold with constraints (see [2,4,11]). Since this model satisfies the bracket generating condition, it follows that the disk can be moved continuously between any two given positions. This example is included here for the sake of completeness.

A disk of radius $R$ rolls on a horizontal plane and is constrained to be vertical at all times. The position of the disk is parameterized by coordinates $(x, y, \theta, \phi) \in \mathbb{R}^2 \times S^1 \times S^1$, where $(x, y)$ may serve as parameters for the center of the disk; $\phi$ is the angle made by some fixed radius on the disk with the vertical; $\theta$ is the angle made by the plane of the disk with the $x$-axis, see Fig. 1(b).
The motion of the disk corresponds to a curve on the space \( M = \mathbb{R}^2 \times S^1 \times S^1 \) that is tangent to the rank 2 distribution defined by the following one-forms

\[
\omega_1 = dx - R \cos \theta \, d\phi, \quad \omega_2 = dy - R \sin \theta \, d\phi.
\]

The following linearly independent vector fields

\[
X_1 = \partial_\theta, \quad X_2 = R(\cos \theta \, \partial_x + \sin \theta \, \partial_y) + \partial_\phi
\]
generate the aforementioned distribution. If consider two other vector fields

\[
X_3 = R(\sin \theta \, \partial_x - \cos \theta \, \partial_y), \quad X_4 = -R(\cos \theta \, \partial_x + \sin \theta \, \partial_y)
\]

the following commutation relations are satisfied

\[
[X_1, X_2] = -[X_1, X_4] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = [X_2, X_4] = 0. \quad (1.1)
\]

Hence by Chow’s theorem we arrive at the following connectivity result, which states that a coin can be rolled continuously between any two given states (see Fig. 1(b)):

**Proposition 1.** Given two points \((x_0, y_0, \phi_0, \theta_0), (x_1, y_1, \phi_1, \theta_1)\) in \(\mathbb{R}^2 \times S^1 \times S^1\), there is at least one piecewise smooth trajectory of the disk that starts with the contact point \((x_0, y_0)\) and initial angles \(\theta_0, \phi_0\) and ends at the contact point \((x_1, y_1)\) having the final angles \(\theta_1, \phi_1\), respectively.

It is worth noting that in this example \(D = \{X_1, X_2\}, D^* = \{X_1, X_2, X_3, X_4\}, \mathcal{H}_x^* = P_D = \mathbb{R}^4\), and the connectivity by horizontal curves is implied by Chow’s theorem. This is not the case for the next example.

### 2. Two-wheel cart

This model has been studied from different perspectives by other authors. For instance, in Zhou and Chirikjian [11] the two-wheel cart is studied assuming noise in the steering. The evolution of the stochastic motion of the cart trajectory is studied by solving the associated Fokker–Planck equation.

In the following we shall emphasize that even if the bracket generating condition does not hold, the connectivity by horizontal curves still holds. Our approach is different in the sense that we show that the system can be moved continuously between any two given states such that the total energy is minimized. The dynamics of the system are described by a Lagrangian with constraints as we shall see below.

Consider a kinematic cart with two equal wheels of radius \(R\) that can roll at different speeds on a plane, so the orientation of the cart might change at any time; see Fig. 2(a). Being connected by an axle of constant length \(L\), the wheels steer together. The state of the cart can be parameterized by the following five coordinates: the coordinates \((x, y) \in \mathbb{R}^2\) of the axle center projection on the \((x, y)\)-plane; the angle \(\theta \in S^1\) made by the plane of the wheels with the \(x\)-axis; the angles \(\phi_1, \phi_2 \in S^1\) made by some fixed radii of the wheels with the vertical direction. These coordinates are not independent, because of the existence of non-holonomic constraints.

Let \((x_i, y_i), i = 1, 2\), be the contact points of the cart wheels with the \((x, y)\)-plane. Since the wheels roll without slipping, we have

\[
dx_i = R \cos \theta \, d\theta, \quad dy_i = R \sin \theta \, d\theta, \quad i = 1, 2.
\]
Fig. 2. (a) A two-wheel cart parameterized by \((x, y, \theta, \phi_1, \phi_2) \in \mathbb{R}^2 \times S^1 \times S^1 \times S^1\); (b) The angle constraint \(L d\theta = -R (d\phi_2 - d\phi_1)\).

Then the midpoint \((x, y)\) satisfies the constraints

\[
\begin{align*}
    dx &= \frac{1}{2} (dx_1 + dx_2) = \frac{R}{2} \cos \theta (d\phi_1 + d\phi_2), \\
    dy &= \frac{1}{2} (dy_1 + dy_2) = \frac{R}{2} \sin \theta (d\phi_1 + d\phi_2).
\end{align*}
\]

(2.1)

Another constraint relates the rotation angle of the axle with the difference of the rotation angles of the wheels, see Fig. 2(b)

\[
L d\theta = R (d\phi_1 - d\phi_2).
\]

(2.2)

Consider the horizontal distribution \(\mathcal{H} = \ker \omega_1 \cap \ker \omega_2 \cap \ker \omega_3\), where

\[
\begin{align*}
    \omega_1 &= dx - \frac{R}{2} \cos \theta (d\phi_1 + d\phi_2), \\
    \omega_2 &= dy - \frac{R}{2} \sin \theta (d\phi_1 + d\phi_2), \\
    \omega_3 &= d\theta - \frac{R}{L} (d\phi_1 - d\phi_2).
\end{align*}
\]

The motion of the cart can be described by a curve \((x(t), y(t), \theta(t), \phi_1(t), \phi_2(t))\) on the space \(M = \mathbb{R}^2 \times S^1 \times S^1 \times S^1\), which is tangent to the rank 2 distribution \(\mathcal{H}\). This way, the motion of the cart can be described by the non-holonomic geometry of \((M, \mathcal{H})\). Consider the linearly independent vector fields \(D = \{X_1, X_2\}\), where

\[
\begin{align*}
    X_1 &= \frac{R}{2} \cos \theta \partial_x + \frac{R}{2} \sin \theta \partial_y + \frac{R}{L} \partial_\theta + \partial_\phi_1, \\
    X_2 &= \frac{2R}{L} \partial_\theta + \partial_\phi_1 - \partial_\phi_2.
\end{align*}
\]

One may check that the distribution \(\mathcal{H}\) is the linear hull of \(D\). Consider two more vector fields

\[
\begin{align*}
    X_3 &= \sin \theta \partial_x - \cos \theta \partial_y, \\
    X_4 &= \cos \theta \partial_x + \sin \theta \partial_y.
\end{align*}
\]
Since we have the following commutation relations

\[ [X_1, X_2] = \frac{R^2}{L} X_3, \quad [X_1, X_3] = \frac{R}{L} X_4, \quad [X_2, X_3] = \frac{2R}{L} X_4, \]

\[ [X_1, X_4] = -\frac{R}{L} X_3, \quad [X_2, X_4] = -\frac{2R}{L} X_3, \]

it follows that \( D^* = \{X_1, X_2, X_3, X_4\} \) and \( X_1, X_2 \) and their iterated brackets generate \( H^* = \text{span}\{X_1, X_2, X_3, X_4\} \), which is a proper subspace of \( T_x M \) at each \( x \in M \). Since \( \dim H_x^* = 4 \), the bracket generating condition does not hold. The set of points where the bracket generating condition fails is the entire manifold \( M \). We may say that the manifold \( (M, H) \) is of infinite step at every point. This is a very different situation from the example exposed in [7, p. 25], where the bracket generating condition fails only on a certain domain. However, \( P_D \) is a distribution of maximal dimension at each point and its maximal integral manifold is the entire space \( M \).

3. Variationally controlled system for the two-wheel cart

In this section, the motion of the two-wheel cart will be studied from the variational point of view. We choose the Lagrangian to be the total kinetic energy of the system plus the non-holonomic constraints

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J (\dot{\phi}_1^2 + \dot{\phi}_2^2) \]

\[ + \mu_1 \left( \dot{x} - \frac{R}{2} \cos(\phi_1 + \phi_2) \right) + \mu_2 \left( \dot{y} - \frac{R}{2} \sin(\phi_1 + \phi_2) \right) \]

\[ + \mu_3 \left( \dot{\theta} - \frac{R}{L} (\phi_1 - \phi_2) \right). \quad (3.1) \]

Here \( m \) denotes the mass of the cart, \( I \) is the moment of inertia of the cart about the symmetry axis perpendicular to the rolling plane and passing through the center of the axle, and \( J \) is the moment of inertia about an axis passing through the centers of the wheels. Since the wheels have equal radii, they also have equal inertia momenta. The Lagrange multipliers are denoted by \( \mu_1, \mu_2 \) and \( \mu_3 \), and are supposed to be functions of \( t \), which is the parameter along the curve. The Euler–Lagrange system of equations for the aforementioned Lagrangian is given by

\[ m \ddot{x} + \mu_1 = 0, \]

\[ m \ddot{y} + \mu_2 = 0, \]

\[ I \ddot{\theta} + \mu_3 = \frac{R}{2} (\mu_1 \sin \theta - \mu_2 \cos \theta) (\dot{\phi}_1 + \dot{\phi}_2), \]

\[ J \dot{\phi}_1 - \frac{R}{2} \left( \mu_1 \cos \theta + \mu_2 \sin \theta + \mu_3 \frac{2}{L} \right) = A, \text{ constant,} \]

\[ J \dot{\phi}_2 - \frac{R}{2} \left( \mu_1 \cos \theta + \mu_2 \sin \theta - \mu_3 \frac{2}{L} \right) = B, \text{ constant.} \]

Substituting \( \omega = \tan^{-1}(\mu_2/\mu_1) \) and \( |\mu| = \sqrt{\mu_1^2 + \mu_2^2} \), the previous system takes the more simple form
m\ddot{x} + \dot{\mu}_1 = 0, \quad (3.2)
m\ddot{y} + \dot{\mu}_2 = 0, \quad (3.3)
I\ddot{\theta} + \dot{\mu}_3 = |\mu| \frac{R}{2} \sin(\theta - \omega)(\dot{\phi}_1 + \dot{\phi}_2), \quad (3.4)
\begin{align*}
J\dot{\phi}_1 - |\mu| \frac{R}{2} \cos(\omega - \theta) - \mu_3 \frac{R}{L} &= A, \quad (3.5) \\
J\dot{\phi}_2 - |\mu| \frac{R}{2} \cos(\omega - \theta) + \mu_3 \frac{R}{L} &= B. \quad (3.6)
\end{align*}

Using the constraint equations (2.1) and integrating Eqs. (3.2)–(3.3) we get
\begin{align*}
\mu_1 &= -m \frac{R}{2} \cos \theta (\dot{\phi}_1 + \dot{\phi}_2) + \alpha, \quad (3.7) \\
\mu_2 &= -m \frac{R}{2} \sin \theta (\dot{\phi}_1 + \dot{\phi}_2) + \beta. \quad (3.8)
\end{align*}

where $\alpha$ and $\beta$ are constants of integration. From here

$$
|\mu|^2 = \mu_1^2 + \mu_2^2 = \frac{m^2 R^2}{4} (\dot{\phi}_1 + \dot{\phi}_2)^2 + mR(\dot{\phi}_1 + \dot{\phi}_2)(\alpha \cos \theta + \beta \sin \theta) + \alpha^2 + \beta^2.
$$

Substituting this into Eqs. (3.4)–(3.6) we obtain the equations that define the dynamics.

A particular case of great importance is obtained when $\alpha = \beta = 0$. This choice yields the non-
holonomic (or the Lagrange–d’Alembert) case. It has been known (see for instance Bloch [1, p. 20])
that the constants of motion $\alpha$ and $\beta$ are not determined by the constraints or initial data. This will
enable us to prove in the next section the global connectivity by geodesics.

In this case, dividing Eqs. (3.7)–(3.8) yields

$$
\frac{\mu_2}{\mu_1} = \tan \theta \quad \Rightarrow \quad \omega = \theta.
$$

Substituting in (3.4) yields

$$
I\ddot{\theta} + \dot{\mu}_3 = 0. \quad (3.9)
$$

On the other hand,

$$
|\mu| = \frac{mR}{2} (\dot{\phi}_1 + \dot{\phi}_2).
$$

Substituting in (3.5)–(3.6) yields

\begin{align*}
J\dot{\phi}_1 - \frac{mR^2}{4} (\dot{\phi}_1 + \dot{\phi}_2) - \mu_3 \frac{R}{L} &= A, \quad (3.10) \\
J\dot{\phi}_2 - \frac{mR^2}{4} (\dot{\phi}_1 + \dot{\phi}_2) + \mu_3 \frac{R}{L} &= B. \quad (3.11)
\end{align*}

Summing up and solving for $\dot{\phi}_1 + \dot{\phi}_2$ yields

$$
\dot{\phi}_1 + \dot{\phi}_2 = k. \quad (3.12)
$$
with \( k = \frac{A + B}{J - \frac{mgR}{2}} \), so \( \phi_1 + \phi_2 = kt + k_0 \). This says that the angular velocity of one wheel with respect to the other wheel is linear in time. Subtracting (3.10)–(3.11) we get

\[
J(\dot{\phi}_1 - \dot{\phi}_2) - \mu_3 \frac{2R}{L} = A - B. \tag{3.13}
\]

Using the constraint (2.2) the previous equation becomes

\[
\frac{JL}{R} \dot{\theta} - \mu_3 \frac{2R}{L} = A - B.
\]

Differentiating yields

\[
\frac{JL}{R} \ddot{\theta} - \mu_3 \frac{2R}{L} = 0.
\]

Using (3.9), eliminating \( \theta \) yields

\[
\mu_3 \left( \frac{JL}{RL} + \frac{2R}{L} \right) = 0 \quad \Rightarrow \quad \mu_3 = \text{constant},
\]

since \( J, L, R \) are positive constants. Then from (3.9) yields \( \ddot{\theta} = 0 \), and hence \( \theta = ct + \theta_0 \), with \( c, \theta_0 \) constants. Substituting in (3.7)–(3.8) implies the following values for the multipliers

\[
\mu_1 = -\frac{mkR}{2} \cos(ct + \theta_0),
\]
\[
\mu_2 = -\frac{mkR}{2} \sin(ct + \theta_0).
\]

Substituting in (3.2)–(3.3) and integrating yields the following parametric equation for the coordinates of the center of mass

\[
x(t) = C_3 + \frac{C_1}{m} t + \frac{kR}{2c} \sin(ct + \theta_0),
\]
\[
y(t) = C_4 + \frac{C_2}{m} t - \frac{kR}{2c} \cos(ct + \theta_0),
\]

where \( C_1, C_2, C_3, C_4 \) are constants of integration. The graph of the trajectory is a drifted circle, see Fig. 3.

Eqs. (3.12) and (3.13) take the form

\[
\dot{\phi}_1 + \dot{\phi}_2 = k,
\]
\[
\dot{\phi}_1 - \dot{\phi}_2 = k_1,
\]

with \( k_1 = \frac{A - B}{J} + \mu_3 \frac{2R}{JL} \), constant. It follows that both \( \phi_1 \) and \( \phi_2 \) are linear functions in \( t \)

\[
\phi_1 = \frac{k + k_1}{2} t + \phi_1(0),
\]
\[
\phi_2 = \frac{k - k_1}{2} t + \phi_2(0).
\]

With this we have completely solved the Euler–Lagrange system of equations.
Fig. 3. The trajectory of the axle center in two cases: (a) \( x(u) = \sin(2u - 2) + u + 1 \), \( y(u) = \cos(2u - 2) + 2u + 5 \); (b) \( x(u) = \sin(10u - 2) + u + 1 \), \( y(u) = \cos(10u - 2) + 2u + 5 \).

4. Global connectivity by geodesics

We shall show that given any two points \( P_0, P \in M \), there is a horizontal curve between \( P_0 \) and \( P \), which is minimizing the total energy.

A solution of the Euler–Lagrange system associated with the Lagrangian (3.1), which satisfies the boundary conditions

\[
\begin{align*}
  x(0) &= x_0, & y(0) &= y_0, & \theta(0) &= \theta_0, & \phi_1(0) &= \phi_{1,0}, & \phi_2(0) &= \phi_{2,0}; \\
  x(T) &= x_T, & y(T) &= y_T, & \theta(T) &= \theta_T, & \phi_1(T) &= \phi_{1,T}, & \phi_2(T) &= \phi_{2,T},
\end{align*}
\]

is a horizontal curve that minimizes the total kinetic energy of the system. This shall be called a non-holonomic geodesic.

The following result states the global connectivity by non-holonomic geodesics. This shows that the cart can be parked starting from any initial position to any final position along an energy minimizing curve.

**Theorem 1.** There is a continuous infinite family of non-holonomic geodesics that satisfies the boundary conditions (4.1)–(4.2).

**Proof.** The explicit formulas for the geodesics components that satisfy the initial conditions (4.1) are

\[
\begin{align*}
  x(t) &= C_3 + \frac{C_1}{m}t + \frac{kr}{2c} \sin(ct + \theta_0), \\
  y(t) &= C_4 + \frac{C_2}{m}t - \frac{kr}{2c} \cos(ct + \theta_0), \\
  \theta(t) &= ct + \theta_0, \\
  \phi_1(t) &= \frac{k + k_1}{2}t + \phi_{1,0}, \\
  \phi_2(t) &= \frac{k - k_1}{2}t + \phi_{2,0}.
\end{align*}
\]

It suffices to show that the final conditions (4.2) determine the constants \( c, k, k_1, C_1, C_2, C_3, C_4 \).
The constant \( c \) is uniquely determined by the condition \( \theta(T) = \theta_T \)

\[
c = \frac{\theta_T - \theta_0}{T}.
\]

From boundary conditions \( \phi_1(T) = \phi_{1,T}, \phi_2(T) = \phi_{2,T} \) we get unique values for the constants \( k \) and \( k_1 \)

\[
k = \frac{1}{T} (\phi_{1,T} + \phi_{2,T} - \phi_{1,0} - \phi_{2,0}),
\]

\[
k_1 = \frac{1}{T} (\phi_{1,T} - \phi_{2,T} + \phi_{2,0} - \phi_{1,0}).
\]

From conditions \( x(T) = x_T, y(T) = y_T \) we can solve to get \( C_3 = C_3(k, c, C_1, T) \) and \( C_4 = C_4(k, c, C_2, T) \) as in the following

\[
C_3 = x_T - \frac{C_1}{m} T - \frac{kR}{2c} \sin(cT + \theta_0),
\]

\[
C_4 = y_T - \frac{C_2}{m} T - \frac{kR}{2c} \cos(cT + \theta_0).
\]

The constants \( C_1, C_2 \) are arbitrary, so there is a 2-parameter family of non-holonomic geodesics that join the initial position with the final position.

It is worth noting that all the non-holonomic geodesics provided by the aforementioned theorem are horizontal curves along which the total kinetic energy

\[
K = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J (\dot{\phi_1}^2 + \dot{\phi_2}^2)
\]

is constant. In fact each of the above terms is constant. Using that along the geodesics we have

\[
\dot{x} = \frac{R}{2} \cos \theta (\dot{\phi_1} + \dot{\phi_2}) = \frac{kR}{2} \cos \theta,
\]

\[
\dot{y} = \frac{R}{2} \sin \theta (\dot{\phi_1} + \dot{\phi_2}) = \frac{kR}{2} \sin \theta,
\]

the constant value of the total kinetic energy is

\[
K = \frac{mk^2R^2}{8} + \frac{1}{2} I c^2 + \frac{J}{4} (k^2 + k_1^2).
\]

This depends on the boundary conditions.

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References