ON HEAT KERNELS OF A CLASS OF DEGENERATE ELLIPTIC OPERATORS

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Abstract. In this article we study the geometry induced by a class of second-order subelliptic operators. This class contains degenerate elliptic and hypoelliptic operators (such as the Grushin operator and the Baouendi-Goulaouic operator). Given any two points in the space, the number of geodesics and the lengths of those geodesics are calculated. We find modified complex action functions and show that the critical values of these functions will recover the lengths of the corresponding geodesics. We also find the volume elements by solving transport equations. Then heat kernels for these operators are obtained.

Key words: subRiemannian geometry, geodesic, Grushin operator, Baouendi-Goulaouic operator, Euler-Lagrange equation, Hamilton-Jacobi equation, heat kernel, action functions, volume elements

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1. Introduction

The purpose of this paper is to study the geometry induced by the following several partial differential operators

\begin{align*}
L_1 &= \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \left( x_k \frac{\partial}{\partial y_k} \right)^2 \\
L_2 &= \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^2 + \frac{x^2}{2} \left\{ \sum_{k=1}^{n} \left( \frac{\partial}{\partial y_k} \right)^2 \right\} \\
L_3 &= \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^2 + \frac{1}{2} \left( x_1^2 + \cdots + x_n^2 \right) \left( \frac{\partial}{\partial y} \right)^2 \\
L_4 &= \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial x_{n+1}} \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \left( x_k \frac{\partial}{\partial y_k} \right)^2.
\end{align*}

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All these operators are closely related to the Grushin operator of step 2:

\[(1.5) \quad L_G = \frac{1}{2}(X_1^2 + X_2^2) = \frac{1}{2}\left( \frac{\partial}{\partial x}\right)^2 + \frac{x^2}{2}\left( \frac{\partial}{\partial y}\right)^2.\]

The operator (1.1) can be realized as a product of Grushin operators (see [13]) defined on the product space \(\mathbb{R}^2 \times \cdots \times \mathbb{R}^2\). The operator (1.2) is a Grushin operator defined on \(\mathbb{R}^{n+1}\) with \(n\) missing directions and the operator (1.3) is a Grushin operator defined on \(\mathbb{R}^{n+1}\) but with only one missing direction. Finally, the operator (1.4) is the well-known example of Baouendi and Goulaouic [1] which mixed the Riemannian and subRiemannian geometries. It is easy to see these operators are elliptic except when \(\{x_k = 0, \, k = 1, \ldots, n\}\). Moreover, one may rewrite these operators as a sum of square of vector fields

\[L_j = \sum_{k=1}^{n} (X_k^2 + Y_k^2), \quad j = 1, 2, 3, 4\]

with \(X_k = \frac{\partial}{\partial x_k}\) and \(Y_k = x_k \frac{\partial}{\partial y_k}, \, k = 1, \ldots, n\) or their variants. We also know that the vector fields \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\) and their first Lie brackets generate the tangent bundle of \(\mathbb{R}^{2n}\). It follows by a theorem of Hörmander [14] that operators \(L_j, \, j = 1, 2, 3, 4\) are hypoelliptic. One may introduce a positive quadratic form \(g\) defined on the span of \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\) such that \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\) are orthonormal, called the subRiemannian metric. Since the vector fields satisfy the bracket generating property, by theorems of Chow [11] and Rashevskii [15], we know that any two points \(P\) and \(Q\) in \(\mathbb{R}^{2n}\) can be connected by a piecewise horizontal curve. This is a curve whose velocity belongs to the distribution defined by span\(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}\). Let \(C(s) = (x_1(s), \ldots, x_n(s), y_1(s), \ldots, y_n(s))\) be a horizontal curve with \(s \in [0, 1]\). The velocity is

\[\dot{C}(s) = (\dot{x}_1(s), \ldots, \dot{x}_n(s), \dot{y}_1(s), \ldots, \dot{y}_n(s))\]

\[= \sum_{k=1}^{n} \left[ \dot{x}_k(s) \frac{\partial}{\partial x_k} + \dot{y}_k(s) \frac{\partial}{\partial y_k} \right] = \sum_{k=1}^{n} \left[ \dot{x}_k(s) X_k + \dot{y}_k(s) Y_k \right].\]

The length of the curve \(C(s)\) is

\[(1.6) \quad \ell(C) = \int_0^1 \sqrt{\sum_{k=1}^{n} \left[ \dot{x}_k^2(s) + \dot{y}_k^2(s) / x_k^2(s) \right]} \, ds.\]

Here we shall give a complete description of the geometry induced by (1.1) to (1.4). When \(n = 1\), we recover the results in [7] and [10].

The fundamental solutions of the operators \(L_j, \, j = 1, 2, 3, 4\) were discussed in [3]. Here we mainly interested in heat kernels. Once we obtain the heat kernels, it is not so difficult to recover results in [3]. In the second part of the paper, we shall use methods developed by Beals,
Gaveau and Greiner in [2] and [4] to give detailed discussion of heat kernels of the diffusion operators $\partial_t - L_j$, $j = 1, 2, 3, 4$. It is known that the heat kernel for $\Delta = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Gaussian:

$$P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4t}}.$$

Given a general second-order elliptic operator $\Delta_X$ in $n$-dimensional Euclidean space, $\Delta_X = \frac{1}{2} \sum_{j=1}^{n} X_j^2 + \text{lower-order terms}$, where $X = \{X_1, \ldots, X_n\}$ is a linearly independent set of vector fields in $\mathbb{R}^n$, the heat kernel takes the form

$$(1.7) \quad P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{d^2(x, x_0)}{4t}} (a_0 + a_1t + a_2t^2 + \cdots).$$

Here $d(x, x_0)$ stands for the Riemannian distance between $x$ and $x_0$ induced by a metric such that vector fields $X_1, \ldots, X_n$ are orthonormal with respect to this metric. The $a_j$'s are functions of $x$ and $x_0$. Note that

$$\frac{\partial}{\partial t} \left( \frac{d^2}{2t} \right) + \frac{1}{2} \sum_{j=1}^{n} \left[ X_j \left( \frac{d^2}{2t} \right) \right]^2 = 0,$$

i.e., $\frac{d^2}{2t}$ is a solution of the Hamilton-Jacobi equation.

From the formula (1.7), it is reasonable to guess the heat kernel for the diffusion operator $\frac{\partial}{\partial t} - L_j$ has the form

$$(1.8) \quad P_t(x, x_0) = \frac{C}{t^d} \int_{\Gamma(x_0)} e^{-\frac{f(x, x_0; \tau)}{t}} V(x, x_0; \tau) d\tau$$

where $\frac{f(x, x_0; \tau)}{t}$ is a solution of the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} \left( \frac{f}{t} \right) + \frac{1}{2} \sum_{j=1}^{m} \left[ X_j \left( \frac{f}{t} \right) \right]^2 = 0.$$

Furthermore, the function $f$ satisfies the generalized Hamilton-Jacobi equation (see [12]):

$$\sum_{j=1}^{m} \tau_j \frac{\partial f}{\partial \tau_j} + \frac{1}{2} \sum_{j=1}^{m} (X_j f)^2 = f.$$

This function plays the same role as $\frac{1}{2} d^2(x, x_0)$ in Riemannian geometry whose critical points will recover all the geodesics. As a consequence, each geodesic has a contribution to the small time asymptotics of the heat kernel. The set $\Gamma(x_0) = \{H = 0\}$ in (1.8) is the characteristic variety of the Hamilton function $H$ of the operator $L_j$ which is a subbundle
in $T^*\mathcal{M}_n$. The weight function $V(x, x_0; \tau)$ is the so-called volume element which is the solution of a transport equation associated to the operator $L_j$. We will give detailed discussion of the action functions $f_j$ and volume elements $V_j$ for operator $L_j, j = 1, 2, 3, 4$ and hence the heat kernels in section 6 of this paper.

2. Hamiltonian mechanics and geometry induced by the operator $L_1$

Let us start with the operator $L_1$. The Hamiltonian of $L_1$ is

$$H(x, y, \xi, \theta) = \frac{1}{2} \sum_{k=1}^{n} \left( \xi_k^2 + x_k^2 \theta_k^2 \right)$$

where the variables $(\xi, \theta) = (\xi_1, \ldots, \xi_n, \theta_1, \ldots, \theta_n)$ are the momenta associated with the coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

A simple calculation yields the Euler-Lagrange equations, for $k = 1, \ldots, n$,

$$\begin{cases}
\dot{y}_k = \theta_k, \\
\dot{x}_k + \frac{\dot{y}_k^2}{x_k} = 0,
\end{cases}$$

(2.1)

where $\theta_1, \ldots, \theta_n$ are constants. All geodesics can be computed explicitly from (2.1).

If $\theta_k = 0$, $k = 1, \ldots, n$, then from (2.1), we have $\dot{y}(s) = 0, \dot{x}(s) =$constant. Hence the geodesic should have the form:

$$\begin{cases}
x(s) = s(x_1 - x_0) + x_0, \\
y(s) = y_0,
\end{cases}$$

(2.2) $s \in [0, 1]$. Moreover, from the first equation of (2.1), one has $\dot{y}_k = \theta_k x_k$. If $\theta_k \neq 0$ and $x_k(s) \neq 0$, then $\dot{y}_k$ is either strictly positive or strictly negative. It follows that $y_k$ is either strictly increasing or strictly decreasing. Hence a geodesic starting from the point $P(x_0, y_0)$ will never return to the hyperplane $\{(x, y) \in \mathbb{R}^{2n} : y = y_0\}$. Hence in order to connect the point $P(x_0, y_0)$ and $Q(x_1, y_0)$, the momentum $\theta_k$ must be zero for all $k$.

Proposition 1. For any two points $P(x_0, y_0)$ and $Q(x_1, y_0)$ on the same hyperplane $\{(x, y) \in \mathbb{R}^{2n} : y = y_0\}$, there is only one geodesic which is a line segment connecting them. The parametric equations of this geodesic are given by (2.2).

Now let us turn to the case when $\theta_k \neq 0$ for all $k$. General solutions of the equation (2.1) with the boundary conditions

(2.3) $x_k(0) = x_{k0}, \quad y_k(0) = y_{k0}, \quad x_k(1) = x_{k1}, \quad y_k(1) = y_{k1}$
are given as follows.

$$\begin{align*}
x_k(s) &= A_k \cos(\theta_k s) + B_k \sin(\theta_k s) \\
y_k(s) &= C_k + \frac{\theta_k}{2} \left[ A_k^2 (s + \frac{\sin(2\theta_k s)}{2\theta_k}) + B_k^2 (s - \frac{\sin(2\theta_k s)}{2\theta_k}) \right] + 2A_kB_k \frac{\sin^2(\theta_k s)}{\theta_k} \\
\end{align*}$$

where

$$\begin{align*}
A_k &= x_{k0} \\
B_k &= \begin{cases} 
\frac{x_{k1} - x_{k0} \cos(\theta_k)}{\sin(\theta_k)} & \text{if } \sin(\theta_k) \neq 0 \\
\text{determined by } y_{k0}, y_{k1}, \text{ and } x_{k0} & \text{if } \sin(\theta_k) = 0 
\end{cases} \\
C_k &= y_{k0}.
\end{align*}$$

It is easy to see that these geodesics are projections of certain bicharacteristics: the solutions of the Hamiltonian system with the boundary condition (2.3)

$$\begin{align*}
\dot{x}_k = H_{\xi_k} = \xi_k \\
\dot{y}_k = H_{\theta_k} = \theta_k x_k^2 \\
\dot{\xi}_k = -H_{x_k} = -\theta_k^2 x_k \\
\dot{\theta}_k = -H_{y_k} = 0,
\end{align*}$$

for $k = 1, \ldots, n$.

A similar system was studied in [2], [4] for the Heisenberg group, in [6] for the step 2($k+1$) model and in [7] for the Grushin operator. The Hamiltonian system is invariant with respect to the following symmetries

$$(x_k, y_k; \theta_k) \to (-x_k, y_k; \theta_k), \quad (x_k, y_k; \theta_k) \to (x_k, -y_k; -\theta_k).$$

These symmetries will send geodesics into geodesics. Without loss of generality, we shall study only the case $x_{k1} > 0$, $y_{k1} - y_{k0} > 0$, $\theta_k > 0$ for $k = 1, \ldots, n$.

Since $H_{y_k} = 0$ for $k = 1, \ldots, n$, it follows that the momentum $\theta_1, \ldots, \theta_n$ are constants which can be considered as Lagrange multipliers. Let $V_k(x_k) = \frac{1}{2} \theta_k^2 x_k^2$. Then $\ddot{x}_k = -V_k'(x_k)$, which can be written also as the conservation of energy law

$$\frac{1}{2} \ddot{x}_k^2 + V_k(x_k) = E_k \quad \text{or} \quad \ddot{x}_k^2 + \theta_k^2 x_k^2 = 2E_k,$$

where $E_k$ is a constant. The energy $E$ depends on the boundary conditions $x_0, x_1, y_0, y_1$ and the constant $\theta_1, \ldots, \theta_n$ can be obtained as follows.

$$\sum_{k=1}^n [\dot{x}_k^2 + \theta_k^2 x_k^2] = 2 \sum_{k=1}^n E_k = 2E,$$

which is a constant along the bicharacteristics.
The second equation of the Hamiltonian system (2.6) yields
\[
y_k^2 = \theta_k^2 x_k^4 \implies \frac{y_k^2}{x_k^2} = \theta_k^2 x_k^2.
\]

Using the conservation law of energy,
\[
\sum_{k=1}^{n} \left[ \dot{x}_k^2 + \frac{y_k^2}{x_k^2} \right] = \sum_{k=1}^{n} \left[ \dot{x}_k^2 + \theta_k^2 x_k^2 \right] = 2E.
\]

It follows from (1.6), one has
\[
\ell(C) = \sqrt{2E}.
\]

In the rest of this section, we shall find explicit formulas for the geodesics connecting the points \( P \) and \( Q \).

When \( \sin(\theta_k) \neq 0 \), then \( y_k(1) = y_{k1} \) for \( k = 1, \ldots, n \) yield
\[
y_{k1} = y_{k0} + \theta_k \left[ \frac{x_{k0}^2}{2} \left( 1 + \frac{\sin(2\theta_k)}{2\theta_k} \right) + \frac{(x_{k1} - x_{k0}\cos(\theta_k))^2}{2\sin^2(\theta_k)} \left( 1 - \frac{\sin(2\theta_k)}{2\theta_k} \right) \right.
\]
\[
+ x_{k0} \left( \frac{x_{k1} - x_{k0}\cos(\theta_k)}{\theta_k \sin(\theta_k)} \right) \sin^2(\theta_k) \left] \right.
\]
\[
= y_{k0} + \frac{x_{k0}^2 + x_{k1}^2}{2\theta_k} \mu(\theta_k) + x_{k0}x_{k1} \left( \sin(\theta_k) - \cos(\theta_k) \mu(\theta_k) \right)
\]

where
\[
\mu(z) = \frac{z}{\sin^2(z)} - \cot(z).
\]

Observe that
\[
\frac{2(y_{k1} - y_{k0})}{x_{k0}^2 + x_{k1}^2} = \mu(\theta_k) + \frac{2x_{k0}x_{k1}}{x_{k0}^2 + x_{k1}^2} \left( \sin(\theta_k) - \cos(\theta_k) \mu(\theta_k) \right)
\]
\[
= \begin{cases} 
(1 - a_k) \mu(\theta_k) + a_k \tilde{\mu}(\frac{\theta_k}{2}), & \text{if } a_k = \frac{2x_{k0}x_{k1}}{x_{k0}^2 + x_{k1}^2} > 0, \\
(1 + a_k) \mu(\theta_k) - a_k \mu(\frac{\theta_k}{2}), & \text{if } a_k = \frac{2x_{k0}x_{k1}}{x_{k0}^2 + x_{k1}^2} < 0,
\end{cases}
\]

where
\[
\tilde{\mu}(z) = \frac{z}{\cos^2(z)} + \tan(z).
\]

It follows that if \( x_{k0}^2 + x_{k1}^2 \neq 0 \), then by a result in [7], there are finitely many solutions \( \theta_{k1}, \ldots, \theta_{kk_N} \) of the equation
\[
\frac{2(y_{k1} - y_{k0})}{x_{k0}^2 + x_{k1}^2} = \begin{cases} 
(1 - a_k) \mu(\theta_k) + a_k \tilde{\mu}(\frac{\theta_k}{2}), & \text{if } a_k > 0, \\
(1 + a_k) \mu(\theta_k) - a_k \mu(\frac{\theta_k}{2}), & \text{if } a_k < 0.
\end{cases}
\]
It follows that there are finitely many solutions $\theta_{11}, \ldots, \theta_{1n}, \theta_{21}, \ldots, \theta_{nn}$ of the system of equations

$$\frac{2(y_k - y_0)}{x_0^2 + x_k^2} = \left[ \mu(\theta_k) + \frac{2x_0x_k}{x_0^2 + x_k^2} \left( \sin(\theta_k) - \cos(\theta_k)\mu(\theta_k) \right) \right], \quad k = 1, \ldots, n.$$  

In other words, given two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ with $x_0^2 + x_k^2 \neq 0$ for $k = 1, \ldots, n$, there are finitely many geodesics connecting $P$ and $Q$. Assume that $\theta_{kk_j}, 1 \leq k_j \leq k_N$, is the solution for the equation

$$\frac{2(y_k - y_0)}{x_0^2 + x_k^2} = \mu(\theta_{kk_j}) + \frac{2x_0x_k}{x_0^2 + x_k^2} \left( \sin(\theta_{kk_j}) - \cos(\theta_{kk_j})\mu(\theta_{kk_j}) \right).$$

Then a direct computation gives us the length of the corresponding geodesic:

$$\ell^2 = \sum_{k=1}^{n} \frac{n}{\sin^2(\theta_{kk_j})} \left[ x_0^2 + x_k^2 - 2x_0x_k \cos(\theta_{kk_j}) \right]$$

$$= \sum_{k=1}^{n} \nu_{a_k}(\theta_{kk_j}) \left[ y_k - y_0 + (x_k - x_0)^2 \right], \quad k_j = 1, 2, \ldots, k_N,$$

where

$$\nu_{a_k}(\theta_k) = \begin{cases} 
\frac{2(1 - a_k \cos(\theta_k))}{(1 - a_k)\mu(\theta_k) + a_k(\bar{\mu}(\theta_k/2) - 2) + 2} \cdot \frac{\theta_k^2}{\sin^2(\theta_k)} & \text{if } a_k = \frac{2x_0x_k}{x_0^2 + x_k^2} > 0, \\
\frac{2(1 - a_k \cos(\theta_k))}{(1 + a_k)\mu(\theta_k) - a_k(\mu(\theta_k/2) + 2) + 2} \cdot \frac{\theta_k^2}{\sin^2(\theta_k)} & \text{if } a_k = \frac{2x_0x_k}{x_0^2 + x_k^2} < 0.
\end{cases}$$

Summarizing the above discussion, one has the following theorem.

**Theorem 1.** Assume that $a_k \neq 0$ and $\sin(\theta_k) \neq 0$ for $k = 1, \ldots, n$. There are finitely many geodesics connecting the points $P(x_0, y_0)$ and $Q(x_1, y_1)$. Their lengths are given by

$$\ell^2 = \sum_{k=1}^{n} \frac{n}{\sin^2(\theta_{kk_j})} \left[ x_0^2 + x_k^2 - 2x_0x_k \cos(\theta_{kk_j}) \right]$$

$$= \sum_{k=1}^{n} \nu_{a_k}(\theta_{kk_j}) \left[ y_k - y_0 + (x_k - x_0)^2 \right], \quad k_j = 1, 2, \ldots, k_N,$$

where $\nu_{a_k}$ is defined by (2.10). Here $\theta_{kk_j}$ are solutions of the equations either

$$\frac{2(y_k - y_0)}{x_0^2 + x_k^2} = (1 - a_k)\mu(\theta_k) + a_k\bar{\mu} \left( \frac{\theta_k}{2} \right), \quad \text{if } a_k > 0,$$

or

$$\frac{2(y_k - y_0)}{x_0^2 + x_k^2} = (1 + a_k)\mu(\theta_k) - a_k\mu \left( \frac{\theta_k}{2} \right), \quad \text{if } a_k < 0.$$
Corollary 1. Given a point \( Q(x_1, y_1) \) with \( x_{k1} \neq 0 \) for \( k = 1, \ldots, n \), there are only finitely many geodesics joining the point \( P(0, y_0) \) and the point \( Q \). Let \( \theta_{k1}, \theta_{k2}, \ldots, \theta_{kk_N} \) be the solutions of the equation
\[
\frac{2(y_{k1} - y_{k0})}{x_{k1}^2} = \mu(\theta_k).
\]
Then the parametric equations of the geodesics are
\[
x_{kk_m}(s) = \sin(\theta_{kk_m} s) x_{k1},
\]
\[
y_{kk_m}(s) = y_{k0} + \frac{x_{k1}^2}{2\sin^2(\theta_{kk_m})} \left( \theta_{kk_m} s - \frac{1}{2} \sin(2\theta_{kk_m} s) \right), \quad k_m = 1, 2, \ldots, k_N.
\]
The length of the corresponding geodesic is
\[
\ell^2 = \sum_{k=1}^{n} \nu(\theta_{kk_m}) \left[ (y_{k1} - y_{k0}) + x_{k1}^2 \right],
\]
where
\[
\nu(z) = \frac{2z^2}{z - \sin(z) \cos(z) + 2\sin^2(z)}.
\]
Proof. The first part of the corollary follows from the formulas (2.4) and (2.5). For the second part, using (2.11) and (2.12) we have
\[
\frac{2(y_{k1} - y_{k0})}{x_{k1}^2} = \mu(\theta_k),
\]
and hence
\[
2 \nu(\theta_{kk_m}) \left( (y_{k1} - y_{k0}) + x_{k1}^2 \right).
\]
Then using \( x_{kk_m}(s) = \frac{\sin(\theta_{kk_m} s)}{\sin(\theta_{kk_m})} x_{k1}, \) (2.7), and (2.13), we have for \( k_m = 1, \ldots, k_N, \)
\[
\ell^2 = 2 \sum_{k=1}^{n} E_k = \sum_{k=1}^{n} \left[ (\dot{x}_k)^2 + \theta_k^2 x_k^2 \right] = \sum_{k=1}^{n} \left[ \left( x_{k1} \frac{\theta_{kk_m}}{\sin(\theta_{kk_m})} \cos(\theta_{kk_m} s) \right)^2 + \theta_{kk_m}^2 \frac{x_{k1}^2}{\sin^2(\theta_{kk_m})} \sin^2(\theta_{kk_m} s) \right] = \sum_{k=1}^{n} \nu(\theta_{kk_m}) \left[ (y_{k1} - y_{k0}) + x_{k1}^2 \right].
\]
The proof of the corollary is therefore complete. \( \square \)

Corollary 2. If \( x_{j1} = x_{j0} = 0 \) for some \( j \) and \( x_{k0} \neq x_{k1} \) for \( k \neq j \) and \( y_0 \neq y_1 \), then there are infinitely many geodesics connecting
The geodesics are given by

\[ P(x_{10}, \ldots, 0, \ldots, x_{n0}, y_0) \] and \[ Q(x_{11}, \ldots, 0, \ldots, x_{n1}, y_1) \]. The parametric equations of the \( j_m \)-th geodesic are given by

(2.14)

\[
\begin{aligned}
x_j(s) &= \sqrt{\frac{2(y_j - y_0)}{j_m \pi}} \sin\left(j_m \pi s\right), \\
x_k(s) &= x_{k0} \cos(\theta_{kkm}s) + \frac{x_{k1} - x_{k0} \cos(\theta_{kkm})}{\sin(\theta_{kkm})} \sin(\theta_{kkm}s), \quad k \neq j \\
y_j(s) &= y_{j0} + (y_{j1} - y_{j0}) \left(s - \frac{\sin(2j_m \pi s)}{2j_m \pi}\right), \quad j_m = 1, 2, 3, \ldots \\
y_k(s) &= y_{k0} + \frac{\theta_{kkm}}{2} \left[\frac{x_{k0}^2 (s + \frac{\sin(2\theta_{kkm}s)}{2\theta_{kkm}})}{\sin^2(\theta_{kkm})} + 2x_{k0}B_{kkm} \left(s - \frac{\sin(2\theta_{kkm}s)}{2\theta_{kkm}}\right)\right], \quad k \neq j,
\end{aligned}
\]

where \( B_{kkm} = \frac{x_{k1} - x_{k0} \cos(\theta_{kkm})}{\sin(\theta_{kkm})} \) and

\[ \ell^2 = 2j_m \pi (y_j - y_0) + \sum_{k=1, k \neq j}^n \frac{\theta_{kkm}^2}{\sin^2(\theta_{kkm})} \left[\frac{x_{k0}^2 + x_{k1}^2 - 2x_{k0}x_{k1} \cos(\theta_{kkm})}{\sin^2(\theta_{kkm})}\right], \]

where \( \ell \) is the length of the corresponding \( j_m \)-th geodesic.

Proof. It remains to calculate lengths of the geodesics. From (2.7), we know that the length of a geodesic \( \ell^2(C) = 2 \sum_{k=1}^n E_k \) where \( E_k \) satisfies the equation \( \dot{x}_k^2 + \theta_k^2 x_k^2 = 2E_k \). Since \( \dot{x}_j(s) = \sqrt{2j_m \pi (y_j - y_0)} \cos(j_m \pi s) \), the result follows immediately. \( \square \)

Remark. If there are two components \( x_{k0}^2 + x_{k1}^2 \) vanish, say \( x_{11}^2 + x_{10}^2 = 0 \) and \( x_{21}^2 + x_{20}^2 = 0 \), then the number of geodesics connecting \( P \) and \( Q \) will be infinite and depends on two parameters \( m_1 \) and \( m_2 \). Therefore, the “infinity” in this case will be “bigger” than the case in Corollary 2. The extremal case will be the following corollary.

Corollary 3. Given the points \( P(0, y_0) \) and \( Q(0, y_1) \), there are infinitely many geodesics connecting \( P \) and \( Q \). The parametric equations of the geodesics are given by, \( 1 \leq k \leq n, \)

(2.15)

\[
\begin{aligned}
x_{km}(s) &= \sqrt{\frac{2(y_{km} - y_0)}{k_m \pi}} \sin\left(k_m \pi s\right), \quad k_m = 1, 2, 3, \ldots \\
y_{km}(s) &= y_{km} + (y_{km} - y_{k0}) \left(s - \frac{\sin(2k_m \pi s)}{2k_m \pi}\right),
\end{aligned}
\]

and

\[ \ell^2_{1, m, \ldots, n_m} = 2\pi \sum_{k=1}^n k_m (y_{km} - y_{k0}), \quad k_m = 1, 2, 3, \ldots, \]
where $\ell_{1_{m-n_m}}$ is the length of the $(1_{m}, \ldots, n_{m})$-th geodesic. For each length $\ell_{1_{m-n_m}}$, there are $2^n$ geodesics connecting the points $P$ and $Q$.

Proof. If $(x_1(s), \ldots, x_n(s), y(s))$ is a solution then $(\pm x_1(s), \ldots, \pm x_n(s), y(s))$ is also a solution because of the Lagrangian symmetries. The result of the corollary follows immediately. \(\square\)

When $\sin(\theta_k) = 0$ for some $k$ and $\sin(\theta_j) \neq 0$ for $j \neq k$, then $\theta_k = m\pi$ and $x_{k1} = \pm x_{k0}$. The parametric equation of the corresponding geodesic has the form

$$
\begin{align*}
  x_k(s) &= x_{k0} \cos(m\pi s) + B_k \sin(m\pi s) \\
  x_j(s) &= x_{j0} \cos(\theta_j s) + \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right) \sin(\theta_j s), \quad \text{for } j \neq k \\
  y_k(s) &= y_{k0} + \frac{m\pi}{2} \left[ x_{k0}^2 (s + \frac{\sin(2m\pi s)}{2m\pi}) + B_k^2 (s - \frac{\sin(2m\pi s)}{2m\pi}) \right] \\
  &\quad + 2x_{k0} B_k \sin^2(m\pi s), \\
  y_j(s) &= y_{j0} + \frac{\theta_j}{2} \left[ x_{j0}^2 (s + \frac{\sin(2\theta_j s)}{2\theta_j}) + \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right)^2 (s - \frac{\sin(2\theta_j s)}{2\theta_j}) \right] \\
  &\quad + 2x_{j0} \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right) \frac{\sin^2(\theta_j s)}{\theta_j}, \quad \text{for } j \neq k
\end{align*}
$$

where $B_k$ is given by

$$
y_{k1} - y_{k0} = \frac{m\pi}{2} \left[ x_{k0}^2 + B_k^2 \right].
$$

It follows that

$$
B_k^2 = \frac{2(y_{k1} - y_{k0})}{m\pi} - x_{k0}^2.
$$

The above makes sense only when

$$
(2.16) \quad \frac{2(y_{k1} - y_{k0})}{m\pi} \geq x_{k0}^2.
$$

Therefore, with given $y_{k0}, y_{k1},$ and $x_{k0}$, there are only finitely many $m$'s such that (2.16) holds. In conclusion, when $x_{k1} = \pm x_{k0}$, besides the geodesics given in Theorem 1, there are finitely many geodesics with $\theta = m\pi$, where $m$ satisfies (2.16). In particular, when $B_k = 0$, there is only one extra geodesic connecting $P$ and $Q$:}

$$
\begin{align*}
  x_k(s) &= x_{k0} \cos(m\pi s) \\
  x_j(s) &= x_{j0} \cos(\theta_j s) + \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right) \sin(\theta_j s), \quad \text{for } j \neq k \\
  y_k(s) &= y_{k0} + \frac{m\pi}{2} \left[ x_{k0}^2 (s + \frac{\sin(2m\pi s)}{2m\pi}) \right] \\
  y_j(s) &= y_{j0} + \frac{\theta_j}{2} \left[ x_{j0}^2 (s + \frac{\sin(2\theta_j s)}{2\theta_j}) + \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right)^2 (s - \frac{\sin(2\theta_j s)}{2\theta_j}) \right] \\
  &\quad + 2x_{j0} \left(\frac{x_{j1} - x_{j0} \cos(\theta_j)}{\sin(\theta_j)}\right) \frac{\sin^2(\theta_j s)}{\theta_j}, \quad \text{for } j \neq k.
\end{align*}
$$
The length $\ell$ of this geodesic satisfies
\[
\ell^2 = m^2 \pi^2 x^2_{k0} + \sum_{j \neq k} \theta_j^2 \left[ \frac{x^2_j + \left( \frac{x_j - x_j \cos(\theta_j)}{\sin(\theta_j)} \right)^2}{x^2_{j0}} \right] - \sum_{j \neq k} \theta_j^2 \left[ \frac{x^2_j + \left( \frac{x_j - x_j \cos(\theta_j)}{\sin(\theta_j)} \right)^2}{x^2_{j0}} \right].
\]

**Theorem 2.** If $x_{k0} = \pm x_{k1} \neq 0$, those $\theta$ satisfying the equation
\[
y_{k1} - y_{k0} = \begin{cases} 
\frac{\mu(\frac{\theta}{2})}{x_{k0}^2} & \text{ if } x_{k0} = x_{k1}, \\
\mu(\frac{\theta}{2}) & \text{ if } x_{k0} = -x_{k1}.
\end{cases}
\]
give finitely many geodesics. They all have different lengths. Furthermore, if
\[
\frac{2(y_{k1} - y_{k0})}{m\pi} \geq x_{k0}^2,
\]
then there are finitely many extra geodesics connecting the points $P$ and $Q$. The length of these geodesic are
\[
2m\pi(y_{k1} - y_{k0}) + \sum_{j \neq k} \theta_j^2 \left[ \frac{x^2_j + \left( \frac{x_j - x_j \cos(\theta_j)}{\sin(\theta_j)} \right)^2}{x^2_{j0}} \right], \quad m = 1, \ldots, N.
\]
In this case, $\theta_m = m\pi$.

### 3. Hamiltonian mechanics and geometry induced by the operator $L_2$

Let us now turn to the operator $L_2$. In this case, the operator has only one “good” direction but with $n$ missing directions. The Hamiltonian becomes
\[
H(x, y, \xi, \theta) = \frac{\xi^2}{2} + \frac{x^2}{2} \left( \sum_{k=1}^{n} \theta_k^2 \right).
\]
The Hamiltonian system is
\[
\begin{cases}
\dot{x} = H_{\xi} = \xi \\
\dot{y}_k = H_{\theta_k} = \theta_k x^2 \\
\dot{\xi} = -H_x = -\left( \sum_{k=1}^{n} \theta_k^2 \right) x \\
\dot{\theta}_k = -H_{y_k} = 0.
\end{cases}
\]

This implies that the Lagrange multipliers $\theta_1, \ldots, \theta_n$ are constants along the bicharacteristics. The Euler-Lagrange equations can be calculated from (3.1) as follows:
\[
\begin{cases}
\frac{\ddot{y}_k}{x^2} = \theta_k, & k = 1, \ldots, n \\
\ddot{x} + \frac{1}{x^2} \left( \sum_{k=1}^{n} \dot{y}_k^2 \right) = 0.
\end{cases}
\]
Denote $\Theta^2 = \sum_{k=1}^{n} \theta_k^2$. The solutions for the system (3.1) with boundary conditions
\[ x(0) = x_0, \quad x(1) = x_1, \quad y(0) = y_0, \quad y(1) = y_1 \]
are given by, $k = 1, \ldots, n$,
\[ y_k(s) = y_{k0} + \theta_k \left\{ \frac{x_0^2}{2} \left( \frac{\sin(2\Theta s)}{2\Theta} \right) + \frac{(x_1 - x_0 \cos(\Theta))^2}{2\sin^2(\Theta)} \left( s - \frac{\sin(2\Theta s)}{2\Theta} \right) \right. \\
+ x_0 \left( \frac{x_1 - x_0 \cos(\Theta)}{\Theta \sin(\Theta)} \right) \sin^2(\Theta) \right\}. \]
The boundary $y(1) = y_1$ gives us for $k = 1, \ldots, n$,
\[ y_k(1) = y_{k0} + \theta_k \left\{ \frac{x_0^2}{2} \left( 1 + \frac{\sin(2\Theta)}{2\Theta} \right) + \frac{(x_1 - x_0 \cos(\Theta))^2}{2\sin^2(\Theta)} \left( 1 - \frac{\sin(2\Theta)}{2\Theta} \right) \right. \\
+ x_0 \left( \frac{x_1 - x_0 \cos(\Theta)}{\Theta \sin(\Theta)} \right) \sin^2(\Theta) \right\} = y_{k0} + \frac{x_0^2 + x_1^2}{2} \mu_k(\Theta) + x_1 x_0 \left[ \frac{\theta_k}{\Theta} \sin(\Theta) - \cos(\Theta) \mu_k(\Theta) \right], \]
where
\[ \mu_k(\Theta) = \frac{\theta_k}{\sin^2(\Theta)} - \frac{\theta_k}{\Theta} \cot(\Theta) = \frac{\theta_k}{\Theta} \left[ \frac{\Theta}{\sin^2(\Theta)} - \cot(\Theta) \right] = \frac{\theta_k}{\Theta} \mu(\Theta), \]
where the function $\mu$ is defined in (2.8). Therefore,
\[ \frac{2(y_{k1} - y_{k0})}{x_0^2 + x_1^2} = \mu_k(\Theta) + \frac{2x_1 x_0}{x_0^2 + x_1^2} \left\{ \frac{\theta_k}{\Theta} \sin(\Theta) - \cos(\Theta) \mu_k(\Theta) \right\} = \frac{\theta_k}{\Theta} \left\{ \mu(\Theta) + \frac{2x_1 x_0}{x_0^2 + x_1^2} \left[ \sin(\Theta) - \cos(\Theta) \mu(\Theta) \right] \right\}. \]
It follows that
\[ \frac{4}{(x_0^2 + x_1^2)^2} \left( \sum_{k=1}^{n} (y_{k1} - y_{k0})^2 \right) = \sum_{k=1}^{n} \theta_k^2 \left\{ \mu(\Theta) + \frac{2x_1 x_0}{x_0^2 + x_1^2} \left[ \sin(\Theta) - \cos(\Theta) \mu(\Theta) \right] \right\}^2. \]
Denote $Y = \left( \sum_{k=1}^{n} (y_{k1} - y_{k0})^2 \right)^{1/2}$. Then the above equation is equivalent to
\[ \frac{2Y}{x_0^2 + x_1^2} \mu(\Theta) + \frac{2x_1 x_0}{x_0^2 + x_1^2} \left[ \sin(\Theta) - \cos(\Theta) \mu(\Theta) \right] = \left\{ (1 - a) \mu(\Theta) + a \mu(\Theta) \right\} \quad \text{if} \quad a = \frac{2x_0 x_1}{x_0^2 + x_1^2} > 0, \]
\[ = \left\{ (1 + a) \mu(\Theta) - a \mu(\Theta) \right\} \quad \text{if} \quad a = \frac{2x_0 x_1}{x_0^2 + x_1^2} < 0 \]
\[ (3.3) \]
Here the function $\tilde{\mu}$ is defined in (2.9). As in the discussion for the operator $L_1$, there are finitely many solutions for the equation (3.3) which denoted by $\Theta_1, \ldots, \Theta_N$.

Using the second equation of the Hamiltonian system (3.1) and the conservation law of energy, one has

$$x^2 + \sum_{k=1}^{n} \left( \frac{y_k^2}{x} \right) = x^2 + \left( \sum_{k=1}^{n} \theta_k^2 \right) x^2 = 2E.$$ 

Then (1.6) yields $\ell(C) = \sqrt{2E}$. Now for the solutions $\Theta_1, \ldots, \Theta_N$ of equation (3.3), we may compute the length of the geodesics as follows.

$$\ell_j^2 = x^2 + \Theta_j^2 x^2 \left[ x_0^2 + \left( \frac{x_1 - x_0 \cos(\Theta_j)}{\sin^2(\Theta_j)} \right)^2 \right]$$

$$= \frac{\Theta_j^2}{\sin^2(\Theta_j)} \left[ x_0^2 + x_1^2 - 2x_0x_1 \cos(\Theta_j) \right]$$

$$= \frac{\Theta_j^2}{\sin^2(\Theta_j)} \left[ 1 - \frac{2x_0x_1}{x_0^2 + x_1^2} \cos(\Theta_j) \right] (x_0^2 + x_1^2).$$

For $a = \frac{2x_0x_1}{x_0^2 + x_1^2} > 0$, we have $\frac{2Y}{x_0^2 + x_1^2} = (1 - a)\mu(\Theta) + a\tilde{\mu}(\Theta)$. Hence,

$$Y + (x_0 - x_1)^2 = Y + x_0^2 + x_1^2 - 2x_0x_1 = (x_0^2 + x_1^2) \left( \frac{Y}{x_0^2 + x_1^2} + 1 - a \right)$$

$$= (x_0^2 + x_1^2) \left\{ \frac{1}{2} - a \mu(\Theta) + \frac{a}{2} \tilde{\mu}(\Theta) + 1 - a \right\}.$$

It follows that

$$\ell_j^2 = \frac{2\Theta_j^2 (1 - a \cos(\Theta_j))}{\sin^2(\Theta_j) \left[ (1 - a)\mu(\Theta_j) + a(\tilde{\mu}(\Theta) - 2) + 2 \right]} [Y + (x_1 - x_0)^2].$$

Similarly, when $a < 0$, the length of the $j$-th geodesic satisfies

$$\ell_j^2 = \frac{2\Theta_j^2 (1 - a \cos(\Theta_j))}{\sin^2(\Theta_j) \left[ (1 + a)\mu(\Theta_j) - a(\mu(\Theta_j) + 2) + 2 \right]} [Y + (x_1 - x_0)^2].$$

Now we have the following theorem.

**Theorem 3.** Assume that $a = \frac{2x_0x_1}{x_0^2 + x_1^2} \neq 0$. The number of geodesics connecting the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is finite which are parametrized by the equations (3.2). Their lengths are given by

$$\ell_j^2 = \begin{cases} 
\frac{2\Theta_j^2 (1 - a \cos(\Theta_j))}{\sin^2(\Theta_j) \left[ (1 - a)\mu(\Theta_j) + a(\tilde{\mu}(\Theta) - 2) + 2 \right]} [Y + (x_1 - x_0)^2], \\
\frac{2\Theta_j^2 (1 - a \cos(\Theta_j))}{\sin^2(\Theta_j) \left[ (1 + a)\mu(\Theta_j) - a(\mu(\Theta_j) + 2) + 2 \right]} [Y + (x_1 - x_0)^2]. 
\end{cases}$$
Here $\Theta_j, j = 1, \ldots, N$, are solutions of the following equation

$$\frac{2Y}{x_0^2 + x_1^2} = \begin{cases} 
(1 - a)\mu(\Theta) + a\tilde{\mu}\left(\frac{\Theta}{2}\right), & a > 0, \\
(1 + a)\mu(\Theta) - a\mu\left(\frac{\Theta}{2}\right), & a < 0.
\end{cases}$$

Similar to the operator $L_1$, there are two extremal cases. We state the results as follows. The proofs of these results are almost the same as Corollaries 1 and 2. We will not repeat them again.

**Corollary 4.** For any two points $P(x_0, y_0)$ and $Q(x_1, y_0)$ on the same hyperplane $\{(x, y) \in \mathbb{R}^{n+1} : y = y_0\}$, there is only one geodesic

$$x(s) = s(x_1 - x_0) + x_0, \quad y(s) = y_0, \quad s \in [0, 1]$$

connecting them.

When $x_0 = x_1 = 0$, then from the Hamiltonian system (3.1) and Theorem 3, one has

$$x(s) = B \sin(m\pi s), \quad y_k(s) = y_{k0} + \theta_k B^2 \left(\frac{s}{2} - \frac{\sin(2m\pi s)}{4m\pi}\right), \quad k = 1, \ldots, n.$$

Since $y_k(1) = y_{k1}$, hence

$$y_{k1} = y_{k0} + \frac{\theta_k}{2} B^2 \Leftrightarrow \theta_k = \frac{2(y_{k1} - y_{k0})}{B^2}.$$

It follows that

$$B = \sqrt{\frac{2Y}{m\pi}}.$$

We have the following corollary.

**Corollary 5.** If $x_1^2 + x_0^2 = 0$ and $y_0 \neq y_1$, then there are infinitely many geodesics connecting $P(0, y_0)$ and $Q(0, y_1)$. The parametric equations of the $m$-th geodesic are given by

$$\begin{cases} 
x_m(s) = \sqrt{\frac{2Y}{m\pi}} \sin(m\pi s), & m = 1, 2, 3, \ldots \\
y_{km}(s) = y_{k0} + (y_{k1} - y_{k0}) \left(\frac{s}{2} - \frac{\sin(2m\pi s)}{2m\pi}\right),
\end{cases}$$

for $k = 1, \ldots, n$. The arc length of the corresponding geodesic is given by

$$\ell_m^2 = 2m\pi Y = 2m\pi \sqrt{\sum_{k=1}^{n} (y_{k1} - y_{k0})^2}.$$

For each arc length $\ell_m$, there are two geodesics connecting the points $P$ and $Q$. 
4. Hamiltonian mechanics and geometry induced by the operator $L_3$

The operator $L_3$ can be considered as the high dimensional Grushin operator. To avoid complicated notations and calculations, we assume $n = 2$. Then the Hamiltonian becomes

$$H(x, y, \xi, \theta) = \frac{1}{2} \left( \sum_{k=1}^{2} \xi_k^2 \right) + \frac{\theta^2}{2} \left( \sum_{k=1}^{2} x_k^2 \right).$$

The Hamiltonian system is

$$\begin{aligned}
\dot{x} &= \xi \\
\dot{\theta} &= 0,
\end{aligned}$$

(4.1)

where $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, and $|x|^2 = \sum_{k=1}^{2} x_k^2$. It is easy to see that the Lagrange multiplier $\theta$ is a constant along the bicharacteristics. The Euler-Lagrange equations can be calculated from (4.1) as follows:

$$\begin{aligned}
\dot{y}|x|^2 &= \theta \\
\ddot{x} + x|x|^{-4}y^2 &= 0.
\end{aligned}$$

We shall find the solution of the system (4.1) with boundary conditions

$$\begin{aligned}
x(0) &= x_0, & x(1) &= x_1, & y(0) &= y_0, & y(1) &= y_1
\end{aligned}$$

(4.2)

as follows.

When $\theta = 0$, from the Hamiltonian system (4.1), we know that $\dot{x} = \dot{\xi} = 0$ and $\dot{y}(s) = 0$. It follows that $y(s) = y_0$, a constant. Moreover, from the second equation of (4.1), one has $\dot{y} = \theta|x|^2$. If $\theta \neq 0$ and $x \neq 0$, then $y$ is either strictly increasing or strictly decreasing. Hence a geodesic starting from the point $P(x_0, y_0)$ will never return to the plane $\{(x, y) \in \mathbb{R}^3 : y = y_0\}$. We also know that $\dot{x}(s) = (C_1, C_2)$, a constant vector. This implies that $x_k(s) = C_k s + D_k$ for $k = 1, 2$. Plugging the boundary conditions (4.2) in the general solutions, we have the following proposition.

**Proposition 2.** For any two points $P(x_0, y_0)$ and $Q(x_1, y_0)$ on the same plane $\{(x, y) \in \mathbb{R}^3 : y = y_0\}$, there is only one geodesic

$$x(s) = s(x_1 - x_0) + x_0, \quad y(s) = y_0, \quad s \in [0, 1]$$

connecting them.
When $\theta \neq 0$, the solution of the system (4.1) with boundary conditions (4.2) are given by

\begin{align*}
\mathbf{x}(s) &= x_0 \cos(\theta s) + \mathbf{B} \sin(\theta s) \\
\xi(s) &= \theta \left( -x_0 \sin(\theta s) + \mathbf{B} \cos(\theta s) \right) \\
y(s) &= y_0 + \frac{\theta}{2} |\mathbf{x}_0|^2 \left[ s + \frac{\sin(2\theta s)}{2\theta} \right] + \langle \mathbf{x}_0, \mathbf{B} \rangle \sin^2(\theta s) \\
&\quad + \frac{\theta}{2} |\mathbf{B}|^2 \left[ s - \frac{\sin(2\theta s)}{2\theta} \right],
\end{align*}

where

$$
\mathbf{B} = \frac{x_1 - x_0 \cos(\theta)}{\sin(\theta)}, \quad \text{if} \quad \sin(\theta) \neq 0.
$$

Let us first make the following observations: We notice that

\begin{equation}
\mathbf{x}(s) = \begin{bmatrix} x_{10} & B_1 \\ x_{20} & B_2 \end{bmatrix} \begin{bmatrix} \cos(\theta s) \\ \sin(\theta s) \end{bmatrix} = [\mathbf{x}_0, \mathbf{B}] \begin{bmatrix} \cos(\theta s) \\ \sin(\theta s) \end{bmatrix}.
\end{equation}

The linear transformation $[\mathbf{x}_0, \mathbf{B}]$ has the polar decomposition $S \circ O$ where $O$ is orthogonal and $S$ is symmetric, hence $S$ can be diagonalized. Now $O$ maps the circle $\{(\cos(\theta s), \sin(\theta s)), \ s \in \mathbb{R}\}$ into a circle, and $S$ maps circles into ellipses (or degenerate to line segments). Hence, when $\det [\mathbf{x}_0, \mathbf{B}] \neq 0$, the geodesics are curves whose projections to the $x_1x_2$-plane are ellipses. This implies they do not intersect $y$-axis.

Now let us turn to the situation when $\det [\mathbf{x}_0, \mathbf{B}] = 0$. This means the vectors $\mathbf{x}_0$ and $\mathbf{B}$ are linearly dependent, they span a one-dimensional linear space, in other words, $\mathbf{x}(s)$ all lie on a line of the $\mathbf{x}$-plane through the origin, namely the line connecting $\mathbf{x}_0, \mathbf{x}_1$. In this case the geodesics are the geodesics of the Grushin operator on the two dimensional plane spanned by the $y$-axis and the line $\overrightarrow{\mathbf{x}_0\mathbf{x}_1}$. So they are curves intersecting the $y$-axis infinitely many times.

We say a geodesic is of type I if it is in a 2-dimensional plane and it intersects $y$-axis infinitely many times. A geodesic is of type II if it does not intersect $y$-axis and its projection onto the $\mathbf{x}$-plane is an ellipse centered at the origin.

Given $P = (x_{10}, x_{20}, y_0) = (\mathbf{x}_0, y_0)$ and $Q = (x_{11}, x_{21}, y_1) = (\mathbf{x}_1, y_1)$ in the space $\mathbb{R}^3$. As the operator $L_3$ is invariant under translation along the $y$-axis, we may assume $y_0 = 0$. We now investigate the geodesics joining $P$ and $Q$. We may divide into following cases:

(i). $\sin \theta \neq 0$. The boundary $y(1) = y_1$ gives us

\begin{align*}
y_1 &= \frac{\theta}{2} |\mathbf{x}_0|^2 \left[ 1 + \frac{\sin(2\theta)}{2\theta} \right] + \langle \mathbf{x}_0, \mathbf{B} \rangle \sin^2(\theta) \\
&\quad + \frac{\theta}{2} |\mathbf{B}|^2 \left[ 1 - \frac{\sin(2\theta)}{2\theta} \right] \\
&= \frac{|x_0|^2 + |x_1|^2}{2} \mu(\theta) + \langle \mathbf{x}_0, \mathbf{x}_1 \rangle \left[ \sin(\theta) - \cos(\theta) \mu(\theta) \right].
\end{align*}
Therefore,
\[
(4.6) \quad \frac{2y_1}{|x_0|^2 + |x_1|^2} = \mu(\theta) + \frac{2 \langle x_0, x_1 \rangle}{|x_0|^2 + |x_1|^2} \left[ \sin(\theta) - \cos(\theta) \mu(\theta) \right].
\]

Similar to the discussion in section 2, one can show that
\[
\sum_{k=1}^{2} \left( \frac{x_k^2}{x_k^2} + \frac{y_k^2}{x_k^2} \right) = \sum_{k=1}^{2} \left( \frac{x_k^2}{x_k^2} + \frac{\theta^2 x_k^2}{x_k^2} \right) = 2 \sum_{k=1}^{2} E_k = 2E,
\]
which is a constant along the bicharacteristics. It follows from (2.7), one has \( \ell(C) = \sqrt{2E} \). Moreover, using the same argument in section 2, one may conclude that there are finitely many \( \theta_k, k = 1, \ldots, N, \) satisfy the formula (4.6) provided \( |x_0|^2 + |x_1|^2 \neq 0 \). From the discussion before, we know that for each \( \theta_k \), the corresponding geodesic will be a curve spiralling along \( (\mathbb{R}, \mathbb{R}) \) and its short axis has length
\[
\frac{\lambda \cos(\theta)}{\sin(\theta)} \sin(\theta)
\]
and its short axis has length
\[
\frac{\lambda \cos(\theta)}{\sin(\theta)} \sin(\theta)
\]
as long as \( x_0 \neq \lambda x_1 \) or \( x_1 \neq \lambda x_0 \). In other words, the line joining \((x_0, y_0)\) and \((x_1, y_1)\) does not intersect the \( y \)-axis.

If \( x_0 = \lambda x_1 \) or \( x_1 = \lambda x_0 \), then
\[
\begin{align*}
  x_1(s) &= x_{10} \left[ \cos(\theta s) + \frac{\lambda \cos(\theta)}{\sin(\theta)} \sin(\theta s) \right] \\
  x_2(s) &= x_{20} \left[ \cos(\theta s) + \frac{\lambda \cos(\theta)}{\sin(\theta)} \sin(\theta s) \right],
\end{align*}
\]
where \( \theta \) satisfies
\[
\frac{2(y_1 - y_0)}{(1 + \lambda^2) |x_0|^2} = \mu(\theta) + \frac{2\lambda}{\lambda^2 + 1} \left[ \sin(\theta) - \mu(\theta) \cos(\theta) \right].
\]

In this case, the geodesics are of type I which are located on the plane spanned by the \( y \)-axis and the line joining \((x_0, y_0)\) and \((x_1, y_1)\).

\( \text{(ii). } \sin \theta = 0, \text{ i.e., } \theta = m\pi \text{ for } m = \pm 1, \pm 2, \ldots \)

\( (1). m = 2k \) for some nonzero integer \( k \). Then by (4.5), one has
\[
x_1 = x_0.
\]

In this case, \( y_1 - y_0 = k\pi |x_0|^2 + k\pi |B|^2 \). Hence,
\[
|B|^2 = \frac{(y_1 - y_0) - k\pi |x_0|^2}{k\pi}.
\]

If \( y_1 - y_0 < k\pi |x_0|^2 \), then there is no geodesic with \( \theta = 2k\pi \), the geodesics connecting \( P \) and \( Q \) should be those obtained in (i). If \( y_1 -
$y_0 = k\pi|x_0|^2$, then $B = 0$ and there is only one geodesic of type $I$ with $\theta = 2k\pi$ as $\text{det } [x_0, B] = 0$. In this case,

\[
\begin{align*}
x(s) &= x_0 \cos(2k\pi s) \\
y(s) &= y_0 + k\pi|x_0|^2\left(s + \frac{\sin(4k\pi s)}{4k\pi}\right)
\end{align*}
\]

and its length $\ell$ satisfies

\[
\ell^2 = 4k^2\pi^2|x_0|^2 = 2k^2\pi^2(|x_0|^2 + |x_1|^2) = 2m\pi(y_1 - y_0).
\]

If $y_1 > k\pi|x_0|^2$, then as $|B|^2 = B_1^2 + B_2^2$, there are infinitely many choices of $B_1$ and $B_2$, i.e., there are infinitely many geodesics.

These geodesics are curves whose projections onto the $x$-plane are ellipses which passing through the points $x_0$ and $-x_0$ with the line segment joining these two points as the degenerate case. The degenerate case is the projection of a type $I$ geodesic.

(2). $m = 2k + 1$ for some $k \in \mathbb{Z}$. Then by (4.5), one has

\[
x_1 = -x_0.
\]

In this case, the projection of $P$ and $Q$ are antipodes and the discussion is almost identical as the case (1).

Now we summarize as follows:

**Theorem 4.** Given $P = (x_0, 0)$ and $Q = (x_1, y_1)$ in $\mathbb{R}^3$, if $\overrightarrow{PQ} \cap y-\text{axis} = \emptyset$, then there are finitely many geodesic of type $II$. They have different lengths

\[
\ell_m^2 = \sum_{k=1}^{2} \frac{\theta_m}{\sin^2(\theta_m)} \left[ x_{k0}^2 + x_{k1}^2 - 2x_{k0}x_{k1}\cos(\theta_m) \right] = \nu(\theta_m) \left[ (y_1 - y_0) + |x_1 - x_0|^2 \right],
\]

where $\theta_m, m = 1, \ldots, N$ are solutions of the equation (4.6).

If $\overrightarrow{PQ} \cap y-\text{axis} \neq \emptyset$, i.e., $x_0 = \lambda x_1$ for some $\lambda \in \mathbb{R}$, then the following holds.

(1). If $\lambda \neq \pm 1$, then there are finitely many geodesics of type $I$ connecting the points $P$ and $Q$, with different lengths. Each geodesic lies on the plane spanned by $\overrightarrow{PQ}$ and the $y$-axis. Again, the $\theta_m, m = 1, \ldots, N$, are determined by the equation (4.6).

(2). If $x_0 = x_1 = 0$, then there are infinitely many geodesics of type $I$ connecting points $P$ and $Q$. The length of the corresponding geodesic is given by

\[
\ell_m = \sqrt{2m\pi|y_1 - y_0|}, \quad m = 1, 2, \ldots.
\]

If $C(s) = (x(s), y(s))$ is a geodesic, then

\[
\begin{align*}
C_\phi(s) &= (e^{i\phi}x(s), y(s)) \\
&= (x_1(s)\cos(\phi) - x_2(s)\sin(\phi), x_1(s)\sin(\phi) + x_2(s)\cos(\phi), y(s))
\end{align*}
\]
is also a geodesic, and $\ell(C_\phi) = \ell(C)$ for all $\phi \in \mathbb{R}$.

(3). If $x_0 = \pm x_1 \neq 0$, those $\theta$ satisfying the equation (4.6) give finitely many geodesics of type I. They all have different lengths. Furthermore, if $y_1 - y_0 = m\pi|x_0|^2$, then there is a type I geodesic connecting the points $P$ and $Q$. The length of this geodesic is $\sqrt{2m\pi(y_1 - y_0)}$. (Note that when $x_0 = x_1$, $m$ is even and when $x_0 = -x_1$, $m$ is odd.) If $|y_1 - y_0| > m\pi|x_0|^2$, then for each $0 < k \leq m$ if $m > 0$ or $m \leq k < 0$ if $m < 0$, there are infinitely many type II geodesics for each $\theta = k\pi$ plus a type I geodesic. They all have the same length $\sqrt{2k\pi(y_1 - y_0)}$ with $\theta = k\pi$.

Remark. Unlike the Grushin operator defined on $\mathbb{R}^2$, there are infinitely many conjugate points of the origin in this case.

5. Hamiltonian mechanics and geometry induced by the operator $L_4$

The operator $L_4$ is the well-known Baouendi-Goulaouic operator which is the sum of a Grushin operator in the $(x_1, \ldots, x_n, y)$ space plus an elliptic operator along the extra direction $x_{n+1}$:

$$L_4 = \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial x_{n+1}} \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \left( x_k \frac{\partial}{\partial y_k} \right)^2.$$ 

To avoid complicated notations and calculations, we assume $n = 1$. Then the Hamiltonian becomes

$$H(x, y, \xi, \theta) = \frac{1}{2} \left( \xi_1^2 + \xi_2^2 + x_1^2\theta^2 \right).$$

The Hamiltonian system is

$$\begin{cases}
\dot{x}_1 = \xi_1 \\
\dot{x}_2 = \xi_2 \\
\dot{y} = \theta x_1^2 \\
\dot{\xi}_1 = -\theta^2 x_1 \\
\dot{\xi}_2 = 0 \\
\dot{\theta} = 0.
\end{cases}$$

(5.1)

From the sixth equation in the Hamiltonian system (5.1), it is easy to see the Lagrange multiplier $\theta$ is a constant along the bicharacteristics. Moreover, the second and the fifth equations of (5.1) imply that $\xi_2 = C_2$ and $x_2(s) = C_2 s + D_2$. Therefore, solution of $x_2(s)$ with boundary conditions

$$x(0) = x_0 = (x_{10}, x_{20}), \quad x(1) = x_1 = (x_{11}, x_{21}), \quad y(0) = y_0, \quad y(1) = y_1$$

is given by

$$x_2(s) = s(x_{21} - x_{20}) + x_{20}. \quad (5.3)$$
Given two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ in $\mathbb{R}^3$, the projection of geodesics connecting $P$ and $Q$ onto the $x_2$-axis are always line segments. The first and the fourth equations of (5.1) imply that $\ddot{x}_1 = \xi_1 = -\theta^2 x_1$. If $\theta = 0$ and $x_1(s) \neq 0$, then
\[
x_1(s) = s(x_{11} - x_{10}) + x_{10}.
\]
Moreover, one has $\dot{y} = 0$ which yields $y(s) = $ constant. This means there is only one geodesic connecting the points $P(x_0, y_0)$ and $Q(x_1, y_0)$ for any fixed number $y_0$.

Now let us turn to the case $\theta \neq 0$. General solutions of the Hamiltonian system (5.1) are given by
\[
\begin{align*}
x_1(s) &= A \cos(\theta s) + B \sin(\theta s) \\
x_2(s) &= (x_{21} - x_{20})s + x_{20} \\
y(s) &= y_0 + \frac{\theta}{2} \left[ A^2 \left( s + \frac{\sin(2\theta s)}{2\theta} \right) + B^2 \left( s - \frac{\sin(2\theta s)}{2\theta} \right) + 2AB \frac{\sin^2(\theta s)}{\theta} \right],
\end{align*}
\]
where
\[
A = x_{10}, \quad B = \begin{cases} 
\frac{x_{11} - x_{10} \cos(\theta)}{\sin(\theta)} & \text{if } \sin(\theta) \neq 0, \\
determined by \ y_1, y_0, \ and \ x_{10} & \text{if } \sin(\theta) = 0.
\end{cases}
\]

For $P(x_{10}, 0, y_0)$ and $Q(x_{11}, 0, y_1)$, the geodesics connecting them are the geodesics for the Grushin operator $\frac{1}{2} \left( \frac{\partial}{\partial x_1} \right)^2 + \frac{1}{2} \left( x_1 \frac{\partial}{\partial y} \right)^2$ on $x_1y$-plane.

When $x_{10} = x_{11} = 0$ and $y_1 - y_0 \neq 0$, we must have $\theta = m\pi$ with $m = 1, 2, \ldots,$ and $B$ is determined by
\[
y_1 = y_0 + \frac{m\pi}{2} B^2 \quad \Leftrightarrow \quad B = \pm \sqrt{\frac{2(y_1 - y_0)}{m\pi}}.
\]

Hence, there are infinitely many geodesics $C_m$, $m = 1, 2, 3, \ldots$, connecting the points $P(0, x_{20}, y_0)$ and $Q(0, x_{21}, y_1)$. The parametric equations of the geodesic $C_m$ are given by
\[
\begin{align*}
x_1(s) &= \sqrt{\frac{2(y_1 - y_0)}{m\pi}} \sin(m\pi s), \quad m = 1, 2, \ldots, \\
x_2(s) &= (x_{21} - x_{20})s + x_{20} \\
y(s) &= y_0 + (y_1 - y_0) \left[ s - \frac{\sin(2m\pi s)}{2m\pi} \right].
\end{align*}
\]

Let $V_1(x) = \frac{1}{2} \theta^2 x_1^2$. Then $\ddot{x}_1 = -V'_1(x_1)$, which can be written also as the conservation law of energy
\[
\frac{1}{2} \ddot{x}_1^2 + V_1(x_1) = E_1 \quad \text{or} \quad \dot{x}_1^2 + \theta^2 x_1^2 = 2E_1,
\]
where $E_1$ is a constant. The second and the fifth equations of the Hamiltonian system (5.1) yield $\ddot{x}_2 = \ddot{\xi}_2 = 0$. Hence there is no external
force along the $x_2$-direction. The energy $E_2$ depends only on the kinetic energy, i.e., $E_2 = \frac{1}{2} x_2^2$. Therefore,

$$\dot{x}_1^2 + \dot{x}_2^2 + \frac{y^2}{x_1^2} = \dot{x}_1^2 + \dot{x}_2^2 + \theta^2 x_1^2 = 2 \sum_{k=1}^{2} E_k = 2E,$$

which is a constant along the bicharacteristics. Following from (2.7), the length of $C_m$ is given by

$$l_m^2 = (x_{21} - x_{20})^2 + 2m\pi(y_1 - y_0).$$

The projection of these curves onto the $x_1y$-plane is shown in Figure 1. The projection onto the $x_2y$-plane is shown in Figure 2 and the projection onto the $x_1x_2$-plane is shown in Figure 3.
Figure 3. Projection of geodesic curves onto the $x_1x_2$-plane, with $m = 1, 2, 3$ (dot, dash, solid, respectively).

In general, if $\sin(\theta) \neq 0$, then solving the first and the second equation of (5.1) with boundary conditions (5.2) yields

\[
\begin{align*}
  x_1(s) &= x_{10} \cos(\theta s) + \left( \frac{x_{11} - x_{10} \cos(\theta)}{\sin(\theta)} \right) \sin(\theta s) \\
  x_2(s) &= (x_{21} - x_{20}) s + x_{20} \\
  y(s) &= y_0 + \frac{\theta}{2} \left[ x_{10}^2 \left( 1 + \frac{\sin(2\theta s)}{2\theta} \right) + \frac{(x_{11} - x_{10} \cos(\theta))^2}{2 \sin^2(\theta)} \left( s - \frac{\sin(2\theta s)}{2\theta} \right) \\
  &+ 2x_{10} \left( \frac{x_{11} - x_{10} \cos(\theta)}{\theta \sin(\theta)} \right) \sin^2(\theta s) \right].
\end{align*}
\]

Since $y(1) = y_1$, the last equation yields

\[
y_1 = y_0 + \frac{\theta}{2} \left[ x_{10}^2 \left( 1 + \frac{\sin(2\theta)}{2\theta} \right) + \frac{(x_{11} - x_{10} \cos(\theta))^2}{2 \sin^2(\theta)} \left( 1 - \frac{\sin(2\theta)}{2\theta} \right) \\
  + 2x_{10} \left( \frac{x_{11} - x_{10} \cos(\theta)}{\theta \sin(\theta)} \right) \sin(\theta) \right] \\
  = y_0 + \frac{x_{10}^2}{2} + \frac{x_{11}^2}{2} \mu(\theta) + x_{10}x_{11} \left( \sin(\theta) - \cos(\theta) \mu(\theta) \right).
\]

It follows that

\[
\frac{2(y_1 - y_0)}{x_{10}^2 + x_{11}^2} = \begin{cases} 
  \mu(\theta) + \frac{2x_{10}x_{11}}{x_{10}^2 + x_{11}^2} \left( \sin(\theta) - \cos(\theta) \mu(\theta) \right) & \text{if } a = \frac{2x_{10}x_{11}}{x_{10}^2 + x_{11}^2} > 0 \\
  (1 - a)\mu(\theta) + a\tilde{\mu} \left( \frac{\theta}{2} \right) & \text{if } a = \frac{2x_{10}x_{11}}{x_{10}^2 + x_{11}^2} < 0 \\
  (1 + a)\mu(\theta) - a\mu \left( \frac{\theta}{2} \right) & \text{if } a = \frac{2x_{10}x_{11}}{x_{10}^2 + x_{11}^2} = 0
\end{cases}
\]

(5.4)

Similar to the discussion in section 2, given two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ in $\mathbb{R}^3$, there are only finitely many geodesics connecting these two points if $x_{10}^2 + x_{11}^2 \neq 0$. 

The length of the \( m \)-th corresponding geodesic connecting \( P \) and \( Q \) is given by
\[
\ell^2_m = \nu(\theta_m) \left[ (y_1 - y_0) + (x_{11} - x_{10})^2 \right] + (x_{21} - x_{20})^2, \quad m = 1, \ldots, N,
\]
where \( \theta_m, m = 1, \ldots, N \) are solutions of the equation (5.4).

When \( \sin(\theta) = 0 \), i.e., \( \theta = m\pi \), we must have \( x_{11} = \pm x_{10} \) (and if \( \theta = 0 \), then \( y_1 \) must equal to \( y_0 \), in addition). The geodesic connecting \( P(x_{10}, x_{20}, y_0) \) and \( Q(x_{11}, x_{21}, y_1) \) has the following parametric equations
\[
\begin{align*}
 x_1(s) &= x_{10} \cos(m\pi s) + B \sin(m\pi s) \\
 x_2(s) &= (x_{21} - x_{20})s + x_{20} \\
 y(s) &= y_0 + \frac{m\pi}{2} \left[ x_{10}^2(s + \frac{\sin(2m\pi s)}{2m\pi}) + B^2(s - \frac{\sin(2m\pi s)}{2m\pi}) + 2x_{10}B\frac{\sin^2(m\pi s)}{2m\pi} \right],
\end{align*}
\]
where \( B \) is given by
\[
y_1 - y_0 = \frac{m\pi}{2} \left[ x_{10}^2 + B^2 \right].
\]
Hence,
\[
B^2 = \frac{2(y_1 - y_0)}{m\pi} - x_{10}^2.
\]
The above makes sense only when
\[
\frac{2(y_1 - y_0)}{m\pi} \geq x_{10}^2. \tag{5.5}
\]
Therefore, with given \( y_1, y_0, \) and \( x_{10} \), there are only finitely many \( m \)'s such that (5.5) holds. In conclusion, when \( x_{11} = \pm x_{10} \), besides the geodesics given by (5.4), there are finitely many geodesics with \( \theta = m\pi \), where \( m \) satisfies (5.5). In particular, when \( B = 0 \), there is only one extra geodesic connecting \( P \) and \( Q \):
\[
\begin{align*}
 x_1(s) &= x_{10} \cos(m\pi s) \\
 x_2(s) &= (x_{21} - x_{20})s + x_{20} \\
 y(s) &= y_0 + \frac{m\pi}{2} \left[ x_{10}^2(s + \frac{\sin(2m\pi s)}{2m\pi}) \right]
\end{align*}
\]
and the length of this geodesic is
\[
\ell^2 = m^2\pi^2 x_{10}^2 + (x_{21} - x_{20})^2 = \frac{m^2\pi^2}{2} (x_{10}^2 + x_{11}^2) + (x_{21} - x_{20})^2.
\]

6. Action functions, volume elements and heat kernels

The classical action function \( S(t) \) is given by
\[
S(t) = \int_0^t \{ \langle \xi, \dot{x} \rangle + \langle \eta, \dot{y} \rangle - H \} \, ds.
\]
It is known that (see [12])
\[
h(x, x_0, y, \eta, t) = \eta(0)y(0) + S(t)
\]
is a solution to the Hamilton-Jacobi equation
\[ \frac{\partial h}{\partial t} + H \left( x, y, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) = 0. \]

And \( h \) has the property:
\[ h(x, y, x_0, \eta, t) = \frac{1}{t} h(x, y, x_0, \eta t, 1). \]

We set the modified action function as
\[ g(x, y, x_0, \lambda) := h(x, x_0, y, \eta t, 1), \]
then \( g \) is a solution of the eikonal equation
\[ \lambda \frac{\partial g}{\partial \lambda} + H \left( x, y, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = g. \]

The modified complex action function is
\[ f(x, x_0, y, \tau) = g(x, x_0, y, -i\tau). \]

As we mentioned in the section 1, we could use \( f(\tau) \) to construct the heat kernel of the associated subelliptic operator in the form of
\[ P_t(x_0, x, y) = C \frac{1}{t^{\alpha}} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau, \]
where \( C \) and \( \alpha \) are two constants, \( \Gamma(x_0) \) is a curve in the complex \( \tau \) space which is the characteristic variety of the Hamiltonian function at the point \( x_0 \). The measure \( V(\tau) \) is the volume element which satisfies the transport equation
\[ \tau \frac{\partial V}{\partial \tau} + (\Delta f - \alpha + d)V = 0, \]
where \( d \) is the number of missing directions. We start by finding the modified complex action function of Grushin operator
\[ L_G = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \right). \]

The associated Hamiltonian is
\[ H(x, y, \xi, \eta) = \frac{1}{2} (\xi^2 + x^2 \eta^2), \]
and the Hamilton system is
\[ \begin{cases} \dot{x} = H_\xi = \xi \\ \dot{y} = H_\eta = \eta x^2 \\ \dot{\xi} = -H_x = -\eta^2 x \\ \dot{\eta} = -H_y = 0. \end{cases} \]

The solution of the Hamilton system is
\[ x(s) = A \sin(\eta(s_0 + s)), \quad y(s) = y_0 + \eta \int_0^s (x(s))^2 ds. \]
Along the bicharacteristics, \((i.e., (x(s), y(s), \xi(s), \eta(s)))\) satisfy the Hamilton system, \(H(s)\) is a constant:

\[
H(s) \equiv H(0) = \frac{1}{2} \eta^2 A^2.
\]

Hence

\[
S(t) = \int_0^t \{\langle \xi, \dot{x} \rangle + \langle \eta, \dot{y} \rangle - H\} \, ds = \int_0^t H(s) \, ds = Ht.
\]

From the Hamilton system, we get

\[
(\dot{x}(s))^2 = \eta^2 A^2 - \eta^2 (x(s))^2,
\]

and integration by parts once, we have

\[
\int_0^t (x(s))^2 \, ds = -\frac{1}{\eta^2} \int_0^t x(s) \dot{x}(s) \, ds
\]

\[
= \left. -\frac{1}{\eta^2} x(s) \dot{x}(s) \right|_{s=0}^{s=t} + \frac{1}{\eta^2} \int_0^t (\eta^2 A^2 - \eta^2 (x(s))^2) \, ds,
\]

which leads to

\[
\eta(y - y_0) = \eta^2 \int_0^t (x(s))^2 \, ds
\]

\[
= \frac{1}{2} \left[ -x(s) \dot{x}(s) \bigg|_{s=0}^{s=t} + \eta^2 A^2 t \right]
\]

\[
= \frac{1}{2} \left[ \eta \sin(\eta s_0) \cos(\eta s_0) - \eta \sin(\eta (s_0 + t)) \cos(\eta (s_0 + t)) + \eta^2 A^2 t \right]
\]

\[
= \frac{1}{2} \left[ \eta^2 (x_0)^2 \frac{\cos(\eta s_0)}{\sin(\eta s_0)} - \eta^2 \frac{\cos(\eta (s_0 + t))}{\sin(\eta (s_0 + t))} + \eta^2 A^2 t \right].
\]

Note that

\[
x^2 \frac{\cos(\eta (s_0 + t))}{\sin(\eta (s_0 + t))} - (x_0)^2 \frac{\cos(\eta s_0)}{\sin(\eta s_0)} - (x^2 + (x_0)^2) \frac{\cos(\eta t)}{\sin(\eta t)}
\]

\[
= x^2 \left( \frac{\cos(\eta (s_0 + t))}{\sin(\eta (s_0 + t))} - \cos(\eta t) \right) - (x_0)^2 \left( \frac{\cos(\eta s_0)}{\sin(\eta s_0)} + \cos(\eta t) \right)
\]

\[
= -x^2 \frac{\sin(\eta s_0)}{\sin(\eta (s_0 + t)) \sin(\eta t)} - (x_0)^2 \frac{\sin(\eta (s_0 + t))}{\sin(\eta s_0) \sin(\eta t)}
\]

\[
= -\frac{2xx_0}{\sin(\eta t)},
\]

where we have made use of \(\frac{x_0}{x} = \frac{\sin(\eta s_0)}{\sin(\eta (s_0 + t))}\). Combining the last two equations, we obtain

\[
\frac{1}{2} \eta^2 A^2 t = \eta(y - y_0) + \frac{\eta}{2 \sin(\eta t)} \left( (x^2 + (x_0)^2) \cos(\eta t) - 2xx_0 \right).
\]
Thus, the modified complex action function is
\[ f(x, x_0, y, \tau) = -i\tau y + \frac{\tau}{2\sinh \tau} \left\{ (x^2 + (x_0)^2) \cosh \tau - 2xx_0 \right\} \]
\[ = -i\tau y + \frac{\tau}{4} \left\{ (x + x_0)^2 \tanh \frac{\tau}{2} + (x - x_0)^2 \coth \frac{\tau}{2} \right\}. \]
Here we may assume that \( y_0 = 0 \) since the Grushin operator \( L_G \) is invariant under translations along the \( y \)-axis. By solving the transport equation
\[ \tau \frac{\partial V}{\partial \tau} + \left( \frac{1}{2}\tau \coth \tau + 1 - \alpha \right) V = 0 \]
with \( \alpha = \frac{3}{2} \), we get the volume element
\[ V(\tau) = \left( \frac{\tau}{\sinh \tau} \right)^{\frac{1}{2}}. \]
The heat kernel of the Grushin operator is
\[ \mathcal{P}_t(x_0, x, y) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Gamma(x_0)} e^{-\frac{f(x)}{t}} V(\tau) d\tau. \]

For detailed discussion of the heat kernel for the Grushin operator, readers can also consult [8] and [9].

6.1. The operator \( L_1 = \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_2^2} + x_1^2 \frac{\partial^2}{\partial y_1^2} + x_2^2 \frac{\partial^2}{\partial y_2^2} \right) \). In this case, the Hamiltonian is
\[ H(x, y, \xi, \eta) = \frac{1}{2} (\xi_1^2 + \xi_2^2 + x_1^2 \eta_1^2 + x_2^2 \eta_2^2). \]
The solution of the associated Hamilton system is
\[ x_1(s) = A_1 \sin(\eta_1 (s_0 + s)), \quad y_1(s) = y_10 + \eta_1 \int_0^s (x_1(s))^2 ds, \]
\[ x_2(s) = A_2 \sin(\eta_2 (s_0 + s)), \quad y_2(s) = y_20 + \eta_2 \int_0^s (x_2(s))^2 ds. \]
Since the operator \( L_1 \) is invariant under translations along the \( y_1 \) and \( y_2 \)-axes, we may assume that \((y_{10}, y_{20}) = (0, 0)\). Therefore, along the bicharacteristics, one has
\[ H(s) \equiv H(0) = \frac{1}{2} (\eta_1^2 A_1^2 + \eta_2^2 A_2^2). \]
In a similar manner as we used in Grushin case, we obtain
\[ \frac{1}{2} \eta_1 A_1^2 t = \eta_1 y_1 + \frac{\eta_1}{2 \sin(\eta_1 t)} ((x_1^2 + (x_{10})^2) \cos(\eta_1 t) - 2x_1 x_{10}), \]
and
\[ \frac{1}{2} \eta_2 A_2^2 t = \eta_2 y_2 + \frac{\eta_2}{2 \sin(\eta_2 t)} ((x_2^2 + (x_{20})^2) \cos(\eta_2 t) - 2x_2 x_{20}). \]
Thus
\[
h(t) = \frac{1}{2} \eta_1 A_1^2 t + \frac{1}{2} \eta_2 A_2^2 t
\]
\[
= \sum_{k=1}^{2} \eta_k y_k + \frac{\eta_k}{2 \sin(\eta_k t)} [(x_k^2 + (x_{k0})^2) \cos(\eta_k t) - 2x_kx_{k0}].
\]

And the modified complex action function is
\[
f(x, x_0, y, \tau) = -i \sum_{k=1}^{2} \tau_k y_k + \frac{\tau_k}{2 \sinh \tau_k} [(x_k^2 + (x_{k0})^2) \cosh \tau_k - 2x_kx_{k0}],
\]
where \( x = (x_1, x_2), \ x_0 = (x_{10}, x_{20}), \ y = (y_1, y_2), \ \tau = (\tau_1, \tau_2). \)

We have essentially the same transport equation as in the Grushin case. With \( \alpha = 3, \) we can solve the volume element as
\[
V(\tau) = \prod_{k=1}^{2} V_k(\tau_k) = \prod_{k=1}^{2} \left( \frac{\tau_k}{\sinh \tau_k} \right)^{1/2}.
\]
The heat kernel for \( L_1 \) is
\[
P_t(x_0, x, y) = \frac{1}{(2\pi t)^{3/2}} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau.
\]

This result can be generalized to any positive integer \( n. \) For the operator
\[
L_{1,n} = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{1}{2} \sum_{k=1}^{n} x_k^2 \frac{\partial^2}{\partial y_k^2},
\]
its heat kernel is
\[
P_t(x_0, x, y) = \frac{1}{(2\pi t)^{3n/2}} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau,
\]
where the modified complex action function is
\[
f(x, x_0, y, \tau) = -i \sum_{k=1}^{n} \tau_k y_k + \frac{\tau_k}{2 \sinh \tau_k} [(x_k^2 + (x_{k0})^2) \cosh \tau_k - 2x_kx_{k0}],
\]
and the volume element is
\[
V(\tau) = \prod_{k=1}^{n} V_k(\tau_k) = \prod_{k=1}^{n} \left( \frac{\tau_k}{\sinh \tau_k} \right)^{1/2}.
\]

\( V(\tau) = \prod_{k=1}^{n} V_k(\tau_k) \) is a solution of the transport equation
\[
\sum_{k=1}^{n} \tau_k \frac{\partial V}{\partial \tau_k} + \left( \frac{1}{2} \sum_{k=1}^{n} \tau_k \coth \tau_k + n - \alpha \right) V = 0,
\]
with \( \alpha = \frac{3n}{2}. \)
Remark: We can rewrite the operator $L_{1,n} = \sum_{k=1}^{n} L_{G_k}$ where $L_{G_k} = \frac{1}{2} \left( \frac{\partial^2}{\partial x_k^2} + x_k^2 \frac{\partial^2}{\partial y^2} \right)$ for $k = 1, \ldots, n$. Moreover, $[L_{G_j}, L_{G_k}] = 0$ for $j \neq k$. Hence

$$e^{-tL_{1,n}} = e^{-t(L_{G_1} + \cdots + L_{G_n})} = \prod_{k=1}^{n} e^{-tL_{G_k}}.$$  

In other words, the heat kernel is just the product of $n$ Grushin heat kernels.

Next, we will deal with the critical values of the modified action function $f(\tau)$. Note that

$$(6.1) \quad g(\lambda) = f(i\lambda) = \sum_{k=1}^{n} \lambda_k y_k + \frac{\lambda_k}{2\sin \lambda_k} ((x_k^2 + (x_{k0})^2) \cos \lambda_k - 2x_k x_{k0}).$$

Thus at the critical point of $f$, we have

$$\frac{\partial f}{\partial \tau_k}(i\lambda) = 0 \Rightarrow \frac{\partial g}{\partial \lambda_k} = 0, \quad k = 1, 2, \ldots, n.$$  

Differentiate (6.1) with respect to $\lambda_k$ yields

$$\frac{\partial g}{\partial \lambda_k} = y_k + \frac{x_k^2 + (x_{k0})^2}{2} \left( \cot \lambda_k - \frac{\lambda_k}{\sin^2 \lambda_k} \right) - x_k x_{k0} \left( \frac{1}{\sin \lambda_k} - \frac{\lambda_k \cos \lambda_k}{\sin^2 \lambda_k} \right)$$

$$= y_k - \frac{x_k^2 + (x_{k0})^2}{2} \mu(\lambda_k) - x_k x_{k0} [\sin \lambda_k - \cos \lambda_k \mu(\lambda_k)].$$

Hence at the critical point of $f$, we have

$$(6.2) \quad \frac{2y_k}{x_k^2 + (x_{k0})^2} = \mu(\lambda_k) + \frac{2x_k x_{k0}}{x_k^2 + (x_{k0})^2} (\sin \lambda_k - \cos \lambda_k \mu(\lambda_k)),$$

for $k = 1, 2, \ldots, n$. This is just the equation we used in section 2 to find all geodesics, and each of the solution ($\lambda_{11}, \cdots, \lambda_{nn}$) corresponds to a geodesic connecting ($x_0, 0$) and ($x, y$). Moreover, since $f(i\lambda) = g(\lambda)$ is a solution to the eikonal equation:

$$\lambda \frac{\partial f}{\partial \lambda} f(i\lambda) + H(x, y, \nabla f) = f(i\lambda),$$

at the critical point $i\lambda = i(\lambda_{11}, \cdots, \lambda_{kk}, \cdots, \lambda_{nn})$ of $f$

$$f(\tau) = f(i\lambda) = H.$$

Noting that $H$ is a constant along the bicharacteristics, and using the relation in (2.7), one has

$$(6.3) \quad f(i\lambda) = H = \ell^2 \frac{2}{2},$$

where $\ell$ is the length of the geodesic joining ($x_0, 0$) and ($x, y$) corresponding to ($\lambda_{11}, \cdots, \lambda_{kk}, \cdots, \lambda_{nn}$).

The critical value of $f$ could be used to derive the small time asymptotics of the heat kernel. Notice that the smallest critical value of $f$ is
where \( d_{CC}^2 \) the Carnot-Carathéodory (subRimannian) distance between \((x_0, 0)\) and \((x, y)\). Other critical values contributes exponentially small than the smallest one in the integral. Hence by a saddle point argument, we have

\[
P_t(x_0, x, y) \sim \frac{1}{(2\pi t)^{\frac{3n}{2}}} e^{-\frac{d_{CC}^2}{2t}} V(\tau) \prod_{k=1}^{n} \sqrt{\frac{2\pi t}{f_{\tau_k}\tau_k}} \sqrt{\frac{\tau_k}{\sin(\tau_k) f_{\tau_k}\tau_k}},
\]

as \( t \to 0^+ \).

**Remark:** By the similar argument, we can also show that the other three operators \( L_2, L_3 \) and \( L_4 \) have the similar properties: the critical points of modified complex action function \( f \) are 1 – 1 correspondence with the geodesics, and the critical values are lengths of corresponding geodesics. And one can also derive the small time asymptotics using the saddle point method.

### 6.2. The operator \( L_2 \)

\( L_2 = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + x^2 \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \right) \). The Hamiltonian for the operator \( L_2 \) is

\[H(x, y, \xi, \eta) = \frac{1}{2} (\xi^2 + x^2 \eta_1^2 + x^2 \eta_2^2).\]

The solution of the associated Hamilton system is

\[x(s) = A \sin(|\eta|(s_0 + s)),\]
\[y_1(s) = y_{10} + \eta_1 \int_0^s (x(s))^2 ds, \quad y_2(s) = y_{20} + \eta_2 \int_0^s (x(s))^2 ds,\]

where \(|\eta| = \sqrt{\eta_1^2 + \eta_2^2}\). Since the operator \( L_2 \) is invariant under translations along the \( y_1 \) and \( y_2 \)-axes, we may assume that \((y_{10}, y_{20}) = (0, 0)\).

Along the bicharacteristics,

\[H(s) \equiv H(0) = \frac{1}{2} |\eta|^2 A^2.\]

In a similar manner as in Grushin case, we obtain

\[\frac{1}{2} |\eta|^2 A^2 t = \sum_{k=1}^{2} \eta_k y_k + \frac{|\eta|}{2 \sin(|\eta| t)} [(x^2 + (x_0)^2) \cos(|\eta| t) - 2xx_0].\]

Thus

\[h(t) = \eta_1 y_{10} + \eta_2 y_{20} + S(t) = \eta_1 y_{10} + \eta_2 y_{20} + \frac{1}{2} |\eta|^2 A^2 t\]
\[= \sum_{k=1}^{2} \eta_k y_k + \frac{|\eta|}{2 \sin(|\eta| t)} [(x^2 + (x_0)^2) \cos(|\eta| t) - 2xx_0].\]
And the modified complex action function is

\[ f(x, x_0, y, \tau) = -i\langle \tau, y \rangle + \frac{|\tau|}{2\sinh|\tau|}((x^2 + (x_0)^2) \cosh|\tau| - 2xx_0), \]

where \( y = (y_1, y_2), \tau = (\tau_1, \tau_2). \)

Note that \( \Delta f \) does not depend on \( y \), we has the similar transport equation as in the Grushin case:

\[ \tau_1 \frac{\partial V}{\partial \tau_1} + \tau_2 \frac{\partial V}{\partial \tau_2} + \left( \frac{1}{2}|\tau| \coth |\tau| + 2 - \alpha \right) V = 0. \]

Assume \( V = V(|\tau|) \), then

\[ \sum_{k=1}^{2} \tau_k \frac{\partial V}{\partial \tau_k} = \sum_{k=1}^{2} \tau_k \frac{\partial V}{|\tau|} \frac{\partial |\tau|}{\partial \tau_k} = \sum_{k=1}^{2} \tau_k^2 \frac{\partial V}{|\tau|} |\tau|^{-1} = |\tau| \frac{\partial V}{\partial |\tau|}. \]

Hence

\[ |\tau| \frac{\partial V}{\partial |\tau|} + \left( \frac{1}{2}|\tau| \coth |\tau| + 2 - \alpha \right) V = 0. \]

With \( \alpha = \frac{5}{2} \), the volume element is

\[ V(\tau) = \left( \frac{|\tau|}{\sinh |\tau|} \right)^{1/2}. \]

The heat kernel for \( L_2 \) is

\[ \mathcal{P}_t(x_0, x, y) = \frac{1}{(2\pi t)^{\frac{5}{2}}} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau. \]

This result can be generalized to any positive integer \( n \). The operator

\[ L_{2,n} = \frac{\partial^2}{\partial x^2} + x^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2} \]

has the heat kernel of the form:

\[ \mathcal{P}_t(x_0, x, y) = \frac{1}{(2\pi t)^{\frac{2n+1}{2}}} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau, \]

where the modified complex action function is

\[ f(x, x_0, y, \tau) = -i\langle \tau, y \rangle + \frac{|\tau|}{2\sinh|\tau|}((x^2 + (x_0)^2) \cosh|\tau| - 2xx_0), \]

and the volume element is

\[ V(\tau) = \left( \frac{|\tau|}{\sinh |\tau|} \right)^{1/2}. \]
6.3. The operator $L_3 = \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + (x_1^2 + x_2^2) \frac{\partial^2}{\partial y^2} \right)$. The Hamiltonian for the operator $L_3$ is

$$H(x, y, \xi, \eta) = \frac{1}{2} (\xi_1^2 + \xi_2^2 + (x_1^2 + x_2^2) \eta^2).$$

The solution of the associated Hamilton system is

$$x_1(s) = A_1 \sin(\eta(s_{10} + s)), \quad x_2(s)X = A_2 \sin(\eta(s_{20} + s)),$$

$$y(s) = y_0 + \eta \int_0^t (x_1(s))^2 + (x_2(s))^2 ds.$$

Along the bicharacteristics,

$$H(s) \equiv H(0) = \frac{1}{2} (\eta^2 A_1^2 + \eta^2 A_2^2).$$

In a similar manner as in Grushin case, we obtain

$$\frac{1}{2} \eta A_1^2 t + \frac{1}{2} \eta A_2^2 t = \eta y + \sum_{k=1}^{2} \frac{\eta}{2 \sin(\eta t)} \left[ (x_k^2 + (x_{k0})^2) \cos(\eta t) - 2x_k x_{k0} \right]$$

since the operator $L_3$ is invariant under translations along the $y$-axis, we may assume that $y_0 = 0$. Thus

$$h(t) = \eta y_0 + S(t) = \eta y_0 + \frac{1}{2} \eta A_1^2 t + \frac{1}{2} \eta A_2^2 t$$

$$= \eta y + \sum_{k=1}^{2} \frac{\eta}{2 \sin(\eta t)} \left[ (x_k^2 + (x_{k0})^2) \cos(\eta t) - 2x_k x_{k0} \right].$$

And the modified complex action function is

$$f(x, x_0, y, \tau) = -i \tau y + \sum_{k=1}^{2} \frac{\tau}{2 \sinh \tau} \left[ (x_k^2 + (x_{k0})^2) \cosh \tau - 2x_k x_{k0} \right],$$

where $x = (x_1, x_2)$, $x_0 = (x_{10}, x_{20})$.

We have the similar transport equation as in the Grushin case:

$$\tau \frac{\partial V}{\partial \tau} + (\tau \coth \tau + 1 - \alpha)V = 0.$$

With $\alpha = 2$, the volume element is

$$V(\tau) = \frac{\tau}{\sinh \tau}.$$

The heat kernel for $L_3$ is

$$\mathcal{P}_t(x_0, x, y) = \frac{1}{(2\pi t)^2} \int_{\Gamma(x_0)} e^{-\frac{f(\tau)}{\tau}} V(\tau) d\tau.$$

This result can be generalized to any positive integer $n$. The heat kernel for

$$L_{3,n} = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{1}{2} \sum_{k=1}^{n} x_k \frac{\partial^2}{\partial y^2}.$$
is
\[
P_t(x_0, x, y) = \frac{1}{(2\pi t)^{\frac{n+2}{2}}} \int_{\Gamma(x_0)} e^{-\frac{f(x)}{t}} V(\tau) d\tau,
\]
where the modified complex action function is
\[
f(x, x_0, y, \tau) = -i\tau y + \frac{\tau}{2\sinh \tau} \sum_{k=1}^{n} \left[ (x_k^2 + (x_k0)^2) \cosh \tau - 2x_k x_{k0}\right]
\]
\[
= -i\tau y + \frac{\tau}{2\sinh \tau} \left( (|x|^2 + |x_0|^2) \cosh \tau - 2(x, x_0)\right),
\]
and the volume element is
\[
V(\tau) = \left(\frac{\tau}{\sinh \tau}\right)^{n/2}.
\]

V(\tau) is the solution of the transport equation
\[
\tau \frac{\partial V}{\partial \tau} + \left( \frac{n}{2} \coth \tau + 1 - \alpha \right) V = 0,
\]
with \(\alpha = \frac{n+2}{2}\).

6.4. The operator \(L_4 = \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + x_1^2 \frac{\partial^2}{\partial y^2} \right)\). In this case, the Hamiltonian is
\[
H(x, y, \xi, \eta) = \frac{1}{2} (\xi_1^2 + \xi_2^2 + x_1^2 \eta^2).
\]

The Solution of the associated Hamilton system is
\[
x_1(s) = A_1 \sin(\eta_1(s_{10} + s)), \quad y(s) = y_0 + \eta \int_0^t (x_1(s))^2 ds,
\]
\[
x_2(s) = \frac{s}{t} (x_2 - x_{20}).
\]

Since the operator \(L_4\) is invariant under translations along the \(x_2\) and \(y_1\)-axes, one may assume that \(x_{20} = 0\) and \(y_0 = 0\). Hence, along the bicharacteristics,
\[
H(s) \equiv H(0) = \frac{1}{2} \eta^2 A_1^2 + \frac{1}{2} \left( \frac{x_2}{t} \right)^2.
\]

In a similar manner as in Grushin case, we obtain
\[
\frac{1}{2} \eta A_1^2 t = \eta y + \frac{\eta}{2\sin(\eta t)}((x_1^2 + (x_{10})^2) \cos(\eta t) - 2x_1 x_{10}).
\]

Thus
\[
h(t) = \eta y_0 + S(t) = \eta y_0 + \frac{1}{2} \eta^2 A_1^2 + \frac{1}{2} \frac{(x_2 - x_{20})^2}{t}
\]
\[
= \eta y + \frac{\eta}{2\sin(\eta t)}((x_1^2 + (x_{10})^2) \cos(\eta t) - 2x_1 x_{10}) + \frac{x_2^2}{2t}.
\]
And the modified complex action function is

\[ f(x, x_0, y, \tau) = -i\tau y + \frac{\tau}{2\sinh \tau}((x_1^2 + (x_{10})^2) \cosh \tau - 2x_1x_{10}) + \frac{x_1^2}{2} \]

where \( x = (x_1, x_2) \), \( x_0 = (x_{10}, 0) \).

For \( k = 1 \), we have essentially the same transport equation as the Grushin case. The volume element is

\[ V(\tau) = \left( \frac{\tau}{\sinh \tau} \right)^{1/2} \]

The heat kernel for \( L_4 \) is

\[ P_t(x_0, x, y) = \frac{1}{(2\pi t)^{3n+2}} \int_{\Gamma(x_0)} e^{\frac{f(\tau)}{2t}} V(\tau) d\tau. \]

This result can be generalized to any positive integer \( n \). The operator

\[ L_{4,n,m} = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \sum_{k=n+1}^{n+m} \frac{\partial^2}{\partial x_k^2} + \frac{1}{2} \sum_{j=1}^{n} x_j^2 \frac{\partial^2}{\partial y_j^2} \]

has the heat kernel of the following form

\[ P_t(x_0, x, y) = \frac{1}{(2\pi t)^{3n+2}} \int_{\Gamma(x_0)} e^{\frac{f(\tau)}{2t}} V(\tau) d\tau, \]

where the modified complex action function is

\[ f(x, x_0, y, \tau) = -i \sum_{j=1}^{n} \tau_j y_j + \frac{\tau_j}{2\sinh \tau_j} \left[ (x_j^2 + (x_{j0})^2) \cosh \tau_j - 2x_jx_{j0} \right] + \sum_{k=n+1}^{n+m} \frac{x_k^2}{2}, \]

and the volume element is

\[ V(\tau) = \prod_{j=1}^{n} V_j(\tau_j) = \prod_{j=1}^{n} \left( \frac{\tau_j}{\sinh \tau_j} \right)^{1/2}. \]

**Remark:** Once again, the operator \( L_{4,n,m} \) can be written as the sum of \( n \) Grushin operator \( \sum_{j=1}^{n} L_{G_j} \) and an elliptic operator \( \Delta_m = \frac{1}{2} \sum_{k=n+1}^{n+m} \frac{\partial^2}{\partial x_k^2} \) defined on \( \mathbb{R}^m \). Since \([L_{G_j}, \Delta_m] = 0\) for \( j = 1, \ldots, n \), hence

\[ P_t(x_1, \ldots, x_n, x_{10}, \ldots, x_{n0}, x_{n+1}, \ldots, x_{n+m}, y_1, \ldots, y_n) \]

\[ = \prod_{j=1}^{n} e^{-tL_{G_j}} \cdot e^{-t\Delta_m} \]

\[ = \frac{e^{-\frac{\sum_{j=1}^{n+m} x_k^2}{2t}}}{(2\pi t)^{3n+2}} \int_{\Gamma(x_0)} e^{i\sum_{j=1}^{n} \tau_j y_j - \frac{\tau_j}{2\sinh \tau_j} \left[ (x_j^2 + (x_{j0})^2) \cosh \tau_j - 2x_jx_{j0} \right]} \prod_{j=1}^{n} \left( \frac{\tau_j}{\sinh \tau_j} \right)^{1/2} d\tau. \]

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